Lecture Notes
on
Astrophysical Radiative Processes

Hsiang-Kuang Chang

Institute of Astronomy
National Tsing Hua University
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Contents

Forewords

This booklet is the lecture notes that I use for the course Astrophysical Radiative Processes, a graduate core course in the Institute of Astronomy, National Tsing Hua University, Hsinchu. It covers concepts and formulae of radiation transfer and various radiation mechanisms. These concepts are fundamental in understanding astrophysical phenomena. The very popular textbook by Rybicki and Lightman (1979 & 2004) is heavily followed in this course. With the intention to make this booklet useful as a handbook to some extent, I have collected many relevant formulae without detailed derivation. In this regard, this booklet can be treated as an extraction version of Rybicki and Lightman's textbook. More fundamental discussion on electromagnetic fields and materials on atomic and molecular line emissions in that textbook are omitted, mainly because of time limitation and my personal preference towards high-energy astrophysics. A parallel discussion of curvature radiation with synchrotron radiation is incorporated, although the curvature radiation finds its application only in strong magnetic fields like that in pulsars' magnetospheres. I strongly suggest readers to have that textbook in hands to cover those omitted materials if wanted and to read more detailed physics discussions. It is also very important to work on the problems at the end of each chapter to improve understanding of the topics covered in that textbook.

Hsiang-Kuang Chang

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Chapter 1

Basics of Radiation Fields and Radiative Transfer

1.1 Introduction

To describe a radiation field, we may denote the energy passing through an area $\mathrm{d}A$ normal to the direction of propagation in time $\mathrm{d}t$, in frequency range $\mathrm{d}\nu$ and in the solid angle $\mathrm{d}\Omega$ as

$$\mathrm{d}E = I_\nu \mathrm{d}\Omega \mathrm{d}\nu \mathrm{d}A \mathrm{d}t \quad , \tag{1.1}$$

where I_ν is the **specific intensity** or the **brightness**. Its unit in the Gaussian system is $[I_\nu] = \mathrm{erg\ s^{-1}\ cm^{-2}\ str^{-1}\ Hz^{-1}}$. The reason to use intensity as the fundamental quantity to describe a radiation field is that it is constant along the way of propagation in vacuum, that is, if there is no interaction with matter along the way. Take an isotropic constant point radiation source as an example. At a certain distance r, the energy passing through a certain area $\mathrm{d}A$ per unit time is inversely proportional to r^2. However, the solid angle $\mathrm{d}\Omega$ subtended by that area $\mathrm{d}A$ is also inversely proportional to r^2. I_ν is therefore constant in r in such a case. The word 'specific' notes that this quantity is referred to as per unit frequency at a certain specific frequency. The mean specific intensity in directions is clearly

$$J_\nu = \frac{1}{4\pi} \oint I_\nu \mathrm{d}\Omega \quad . \tag{1.2}$$

The flux, or more precisely, the **specific flux or flux density** in a certain direction $\hat{n}$ is

$$F_\nu = \oint I_\nu \cos\theta \mathrm{d}\Omega \ , \tag{1.3}$$

where $\cos\theta = \hat{n} \cdot \hat{\Omega}$ and $\hat{\Omega}$ refers to the direction of the radiation being integrated. The unit used for the flux density is called Jansky (Jy), and

1 Jy $= 10^{-23}$ erg s^{-1} cm^{-2} Hz^{-1}.

The momentum flux in the direction $\hat{n}$ is then, multiplying one more factor of $\cos\theta$ to account for the momentum component in the direction $\hat{n}$ and considering $p = E/c$ for photons,

$$p_\nu = \frac{1}{c} \oint I_\nu \cos^2\theta \mathrm{d}\Omega \ . \tag{1.4}$$

The radiation pressure (per unit frequency) of this radiation field, for a totally reflecting material, is

$$P_\nu = \frac{2}{c} \int I_\nu \cos^2\theta \mathrm{d}\Omega \ , \tag{1.5}$$

where the integration over θ is between 0 and $\frac{\pi}{2}$ only. This can be understood by considering that pressure is simply a momentum flux. For an isotropic radiation field, we have $P_\nu = \frac{4\pi I_\nu}{3c} = \frac{1}{3} u_\nu$, with $u_\nu \mathrm{d}\nu$ being the energy density.

1.2 Radiative transfer

Intensity may change due to emission, absorption, and scattering along the way of propagation. Consider the energy emitted by a volume $\mathrm{d}V$ into the solid angle $\mathrm{d}\Omega$ in time $\mathrm{d}t$ as

$$\mathrm{d}E_\mathrm{e} = j\mathrm{d}V\mathrm{d}\Omega\mathrm{d}t = j_\nu \mathrm{d}s\mathrm{d}A\mathrm{d}\Omega\mathrm{d}t\mathrm{d}\nu \ ,$$

we have

$$\mathrm{d}I_\nu = j_\nu \mathrm{d}s \ . \tag{1.6}$$

j_ν thus defined is called the **emission coefficient**. We usually define the **absorption coefficient** α_ν as

$$\mathrm{d}I_\nu = -\alpha_\nu I_\nu \mathrm{d}s \ . \tag{1.7}$$

In terms of **opacity** κ_ν or the cross section σ_ν, we have

$$\alpha_\nu = \rho\kappa_\nu = n\sigma_\nu \ , \tag{1.8}$$

where n is the number density and ρ is the mass density of the matter involved in the corresponding absorption process. The opacity κ_ν is also called the mass absorption coefficient. The cross-section expression is valid for the case that only one process is involved.

The radiative transfer equation, without scattering for the moment, reads

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu I_\nu + j_\nu \ . \tag{1.9}$$

For the case of emission only, $\alpha_\nu = 0$,

$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu(s')\mathrm{d}s' \ , \tag{1.10}$$

and for absorption only, $j_\nu = 0$,

$$I_\nu(s) = I_\nu(s_0)\exp(-\int_{s_0}^{s} \alpha_\nu(s')\mathrm{d}s') \ . \tag{1.11}$$

It is convenient to define the **optical depth** τ_ν as

$$\mathrm{d}\tau_\nu = \alpha_\nu \mathrm{d}s \ . \tag{1.12}$$

We then have

$$\tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu(s')\mathrm{d}s' \tag{1.13}$$

with $\tau_\nu(s_0) = 0$. When a system has $\tau_\nu \gg 1$, it is optically thick (opaque), and when $\tau_\nu \ll 1$, it is optically thin (transparent).

We define the **source function** S_ν as $S_\nu = j_\nu/\alpha_\nu$, and then the radiative transfer equation becomes

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = -I_\nu + S_\nu \ , \tag{1.14}$$

whose solution is

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(\tau'_\nu)e^{-(\tau_\nu-\tau'_\nu)}\mathrm{d}\tau'_\nu \ . \tag{1.15}$$

If S_ν is a constant, the solution is

$$\begin{aligned} I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu(1-e^{-\tau_\nu}) \\ &= S_\nu + e^{-\tau_\nu}(I_\nu(0) - S_\nu) \ . \end{aligned} \tag{1.16}$$

We can see that I_ν approaches S_ν when $\tau_\nu \to \infty$. This can also be seen from Eq.(1.14), without assuming S_ν being constant, that when $I_\nu > S_\nu$ we will have $\mathrm{d}I_\nu/\mathrm{d}\tau_\nu < 0$ and vice versa. The specific intensity approaches the source function. In this sense it is a relaxation process. On the other hand, for $\tau_\nu \ll 1$, besides the contribution from $I_\nu(0)e^{-\tau_\nu}$, I_ν is approximately equal to $\tau_\nu S_\nu$.

The **mean free path** ℓ_ν is the average distance that a photon can travel before being absorbed. To associate ℓ_ν with the absorption coefficient α_ν, let's consider the mean optical depth. From Eq.(1.11) we see that the survival probability of a photon after traveling an optical depth τ_ν is $e^{-\tau_\nu}$. The mean optical depth is then

$$\langle\tau_\nu\rangle = \int_0^\infty \tau_\nu e^{-\tau_\nu}\mathrm{d}\tau_\nu = 1 \ . \tag{1.17}$$

This is why an optical depth of unity is usually taken to be the 'visible' depth, or the boundary of being opaque or transparent. In a homogeneous medium, we may have $\langle\tau_\nu\rangle = \alpha_\nu\ell_\nu$ (Eq.(1.12)), which is equal to 1. Therefore

$$\ell_\nu = \frac{1}{\alpha_\nu} = \frac{1}{n\sigma_\nu} = \frac{1}{\rho\kappa_\nu} \ . \tag{1.18}$$

We may also interpret this ℓ_ν as a local mean free path for any media.

1.3 Thermal radiation and Kirchhoff's law

Thermal radiation is radiation emitted by matter in thermal equilibrium, and **blackbody radiation** is radiation which is itself in thermal equilibrium. The specific intensity of a blackbody radiation is the Planck law, that is,

$$B_\nu(T) = \frac{2\nu^2}{c^2}\frac{h\nu}{e^{h\nu/kT}-1} \ . \tag{1.19}$$

Kirchhoff's law for thermal radiation is that the source function of matter in thermal equilibrium is the Planck function:

$$S_\nu \equiv \frac{j_\nu}{\alpha_\nu} = B_\nu(T) \ . \tag{1.20}$$

This can be seen by considering a blob of matter in thermal equilibrium immersed in a blackbody radiation field of the same temperature. The source function, which is the ratio of emissivity and the absorption coefficient, is an intrinsic, thermodynamic property of the matter. In this example, because of thermal equilibrium, the emissivity of this matter will be equal to what it absorbs from the ambient, that is, $j_\nu = \alpha_\nu B_\nu$. Kirchhoff's law relates j_ν and α_ν with the temperature of the matter. In many applications, thermal equilibrium is assumed to be locally true. This is called the assumption of **local thermal equilibrium (LTE)**. Sometimes distinction in the terminology is made in the following way: When the source function does not include scattering, it is called weak LTE. When scattering is included in the source function, it is modest LTE. When the specific intensity is assumed to be equal to the Planck function, i.e., $I_\nu = B_\nu$, it is called strong LTE.

One thing to note is that the Planck intensity is

$$B(T) = \int_0^\infty B_\nu(T)\mathrm{d}\nu = \frac{2h}{c^2}(\frac{kT}{h})^4 \int_0^\infty \frac{x^3\mathrm{d}x}{e^x - 1} = \frac{2\pi^4 k^4}{15c^2h^3}T^4 \ , \tag{1.21}$$

where the value $\pi^4/15$ of the integral over x is employed. Considering the flux F from the surface of a blackbody,

$$F = \int_{\theta=0}^{\theta=\frac{\pi}{2}} B\cos\theta\mathrm{d}\Omega = \pi B =: \sigma T^4 \ , \tag{1.22}$$

we have the **Stefan-Boltzman constant** σ to be $\frac{2\pi^5k^4}{15c^2h^3}$, which is 5.67×10^{-5} erg cm^{-2} deg^{-4} s^{-1}.

One way of characterizing the brightness at a certain frequency is to assign a **brightness temperature** T_b so that

$$I_\nu = B_\nu(T_\mathrm{b}) \tag{1.23}$$

at that frequency. With the LTE assumption, we have

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = -I_\nu + B_\nu(T) \ . \tag{1.24}$$

In the regime that $h\nu \ll kT_\mathrm{b}$, we have

$$I_\nu = \frac{2\nu^2}{c^2}kT_\mathrm{b} \tag{1.25}$$

and therefore

$$\frac{\mathrm{d}T_{\mathrm{b}}}{\mathrm{d}\tau_\nu} = -T_{\mathrm{b}} + T \ . \tag{1.26}$$

We can see that, when $\tau_\nu \gg 1$, I_ν approaches $B_\nu(T)$ and T_{b} approaches T, similar to what we get for Eq. (1.14).

1.4 The Einstein coefficients

Kirchhoff's law implies some relation between emission and absorption at a microscopic level. Let's consider a two-level atomic system. Level 1 is at energy E with statistical weight g_1 and level 2 is at energy $E + h\nu_0$ with statistical weight g_2. [1] Define the Einstein A-coefficient A_{21} to be the transition probability per unit time for **spontaneous emission**, the Einstein B-coefficient B_{12} with $B_{12}\bar{J}_\nu$ being that for **absorption**, and another Einstein B-coefficient B_{21} with $B_{21}\bar{J}_\nu$ being that for **stimulated emission**. $\bar{J}_\nu$ here is a frequency-weighted average of the average specific intensity, defined as

$$\bar{J}_\nu(\nu_0) = \int_0^\infty J_\nu \phi(\nu;\nu_0)\mathrm{d}\nu \ , \tag{1.27}$$

where $\phi(\nu;\nu_0)$ is the line shape function sharply peaked at ν_0 and is normalized as $\int \phi(\nu;\nu_0)\mathrm{d}\nu = 1$. The inclusion of the stimulated emission process is the key to find relations among Einstein coefficients. The consideration of the line shape function is not essential here but is needed when relating j_ν and α_ν with Einstein coefficients.

In thermal equilibrium, we have

$$n_1 B_{12}\bar{J}_\nu = n_2 A_{21} + n_2 B_{21}\bar{J}_\nu \ , \tag{1.28}$$

where n_1 and n_2 are the number density of atoms at level 1 and level 2 respectively. Since

$$\frac{n_1}{n_2} = \frac{g_1}{g_2}\frac{e^{-E/kT}}{e^{-(E+h\nu_0)/kT}} = \frac{g_1}{g_2}e^{h\nu_0/kT} \ , \tag{1.29}$$

[1]The statistical weight is the degeneracy, or, the number of possible quantum states of a certain energy level.

and

$$\bar{J}_\nu = \frac{A_{21}/B_{21}}{(n_1 B_{12}/n_2 B_{21}) - 1} \ ,$$

we have

$$\bar{J}_\nu = \frac{(A_{21}/B_{21})}{(g_1 B_{12}/g_2 B_{21}) e^{h\nu_0/kT} - 1} \ . \tag{1.30}$$

With $J_\nu = B_\nu$ in thermal equilibrium, we expect $\bar{J}_\nu(\nu_0) = B_\nu(\nu_0)$ since $\phi(\nu;\nu_0)$ is sharply peaked at ν_0. This can only be achieved by assigning the following relations:

$$g_1 B_{12} = g_2 B_{21} \ , \tag{1.31}$$

and

$$A_{21} = \frac{2\nu_0^2}{c^2} h\nu_0 B_{21} \ . \tag{1.32}$$

The above two equations are the so-called **detailed balance relations**. Since Einstein coefficients, A_{21}, B_{21}, and B_{12}, are intrinsic properties of an atom, their values do not depend on whether the system is in thermal equilibrium or not. The detailed balance relations are therefore always valid. They relate emission and absorption at a microscopic level and can be used to derive an extension of Kirchhoff's law to include non-thermal emissions.

The energy spontaneously emitted from such a two-level system in a differential range of $\mathrm{d}V\mathrm{d}t\mathrm{d}\Omega\mathrm{d}\nu$ is

$$h\nu n_2 A_{21} \mathrm{d}V\mathrm{d}t \frac{\mathrm{d}\Omega}{4\pi} \phi(\nu)\mathrm{d}\nu = j_\nu \mathrm{d}V\mathrm{d}t\mathrm{d}\Omega\mathrm{d}\nu \ . \tag{1.33}$$

At the right hand side of the above equation, a macro expression is employed. We therefore have the emission coefficient to be

$$j_\nu = \frac{h\nu\phi(\nu)}{4\pi} n_2 A_{21} \ . \tag{1.34}$$

For the absorption coefficient, let's consider the energy absorbed in $\mathrm{d}V\mathrm{d}t$, which is

$$h\nu n_1 \mathrm{d}V B_{12} \bar{J}_\nu \mathrm{d}t = h\nu n_1 \mathrm{d}V B_{12} \left(\int \left(\frac{1}{4\pi}\int I_\nu \mathrm{d}\Omega\right)\phi(\nu)\mathrm{d}\nu\right)\mathrm{d}t \ . \tag{1.35}$$

The energy absorbed out of a beam in $\mathrm{d}V\mathrm{d}t\mathrm{d}\Omega\mathrm{d}\nu$ is then equal to

$$h\nu n_1 \mathrm{d}V B_{12}\frac{1}{4\pi}I_\nu \mathrm{d}\Omega\phi(\nu)\mathrm{d}\nu\mathrm{d}t = \alpha_\nu I_\nu \mathrm{d}s\mathrm{d}A\mathrm{d}t\mathrm{d}\Omega\mathrm{d}\nu \ . \tag{1.36}$$

Again at the right hand side of the above equation a macro expression is employed. The absorption coefficient, uncorrected for stimulated emission, is then

$$\alpha_\nu = \frac{h\nu\phi(\nu)}{4\pi}n_1 B_{12} \ . \tag{1.37}$$

The stimulated emission is better treated as a 'negative absorption'. In the same manner as the above discussion, we have the absorption coefficient as

$$\alpha_\nu = \frac{h\nu\phi(\nu)}{4\pi}(n_1 B_{12} - n_2 B_{21}) \ . \tag{1.38}$$

We can now write down the equation of radiation transfer with Einstein coefficients, which is

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\frac{h\nu\phi(\nu)}{4\pi}(n_1 B_{12} - n_2 B_{21})I_\nu + \frac{h\nu\phi(\nu)}{4\pi}n_2 A_{21} \ , \tag{1.39}$$

and we note that the source fundtion is

$$S_\nu = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} \ . \tag{1.40}$$

From the detailed balance relations, Eq.(1.31) and Eq.(1.32), we have

$$\alpha_\nu = \frac{h\nu\phi(\nu)}{4\pi}n_1 B_{12}(1 - \frac{g_1 n_2}{g_2 n_1}) \tag{1.41}$$

and

$$S_\nu = \frac{2\nu^2}{c^2}\frac{h\nu}{(\frac{g_2 n_1}{g_1 n_2} - 1)} \ . \tag{1.42}$$

One sees that the source function in fact does not depend on Einstein coefficients, and Eq.(1.42) is called **the generalized Kirchhoff's law**, which applies to all situations, including non-thermal cases.

In thermal equilibrium, noting that $\frac{n_1}{n_2} = \frac{g_1}{g_2}e^{(h\nu/kT)}$, we have

$$\alpha_\nu = \frac{h\nu\phi(\nu)}{4\pi}n_1 B_{12}(1 - e^{-h\nu/kT}) \tag{1.43}$$

and $S_\nu = B_\nu$. This can be regarded as a proof of Kirchhoff's law for thermal radiation. We can also see that α_ν is always positive in such a case. To have a negative α_ν, like in the cases of laser or maser, we need to have an inverted population, that is,

$$\frac{n_1}{g_1} < \frac{n_2}{g_2} \ . \tag{1.44}$$

1.5 Scattering in radiative transfer

For simplicity, let's consider only isotropic and coherent scattering. The emission coefficient for isotropic, coherent scattering can be found by equating the absorbed and emitted power due to scattering, that is,

$$\int \sigma_\nu I_\nu \mathrm{d}\Omega = 4\pi j_\nu \ , \tag{1.45}$$

where σ_ν is the **scattering coefficient**. We therefore have, for scattering-only processes, the emission coefficient as

$$j_\nu = \sigma_\nu J_\nu \ , \tag{1.46}$$

the source function as

$$S_\nu = \frac{j_\nu}{\sigma_\nu} = J_\nu \tag{1.47}$$

and the radiative transfer equation as

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\sigma_\nu (I_\nu - J_\nu) \ . \tag{1.48}$$

Now let's put absorption and scattering together. The equation then reads

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu I_\nu + j_\nu - \sigma_\nu I_\nu + \sigma_\nu J_\nu \ , \tag{1.49}$$

which can be turned into

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -(\alpha_\nu + \sigma_\nu)(I_\nu - S_\nu) \ , \tag{1.50}$$

by defining the source function as

$$S_\nu = \frac{j_\nu + \sigma_\nu J_\nu}{\alpha_\nu + \sigma_\nu} \quad . \tag{1.51}$$

When $j_\nu/\alpha_\nu = B_\nu$ is used we will have

$$S_\nu = \frac{\alpha_\nu B_\nu + \sigma_\nu J_\nu}{\alpha_\nu + \sigma_\nu} \quad . \tag{1.52}$$

The term $(\alpha_\nu + \sigma_\nu)$ is the total absorption coefficient, which is also called the **extinction coefficient**.

Let's now consider the length between the locations of a photon being emitted and absorbed. The mean free path for a photon to travel before being absorbed or scattered is

$$\ell = \frac{1}{\alpha_\nu + \sigma_\nu} \quad . \tag{1.53}$$

During the random walk of scattering, the probability for a photon being absorbed after travelling a free path is

$$\epsilon = \frac{\alpha_\nu}{\alpha_\nu + \sigma_\nu} \quad . \tag{1.54}$$

If N is the number of paths taken by a photon before being absorbed, that is, $N\epsilon = 1$, the length between the locations of a photon being emitted and absorbed will be

$$\ell_* = \sqrt{N}\ell = \frac{\ell}{\sqrt{\epsilon}} = \frac{1}{\sqrt{\alpha_\nu(\alpha_\nu + \sigma_\nu)}} \quad . \tag{1.55}$$

This length, ℓ_*, is called the **diffusion length** or the **effective mean free path**. For a homogeneous medium of size L, we may also express the above equation in terms of optical thickness with $\tau_* := L/\ell_*$, $\tau_a := \alpha_\nu L$, and $\tau_s := \sigma_\nu L$, as the following:

$$\tau_* = \sqrt{\tau_a(\tau_a + \tau_s)} \quad . \tag{1.56}$$

The τ_* here is called the **effective optical depth** or **effective optical thickness**. If $\tau_* \ll 1$, the medium is called effectively thin or translucent. Most photons emitted by the medium can escape out of the medium. If, on the other hand, $\tau_* \gg 1$, the medium is called effectively thick. Photons at depths larger than the effective mean free path will be mostly absorbed before escaping out of the medium. Therefore, for thermally emitted photons at large effective optical depth, radiation tends to become in thermal equilibrium with matter, and one can expect $I_\nu \rightsquigarrow B_\nu$ and $S_\nu \to B_\nu$. The latter is Kirchhoff's law of the (modest) LTE assumption, which includes scattering. For this reason, ℓ_* is also called the **thermalization length**. The specific intensity, which is usually the solution that we are looking for, is not as close as the source function to the Planck function. These points are further elaborated in the next section.

1.6 Radiative diffusion

We have demonstrated that in a homogeneous medium the source function S_ν approaches B_ν at large effective optical depth. Media are in general not homogeneous, but very often, such as in the deep interior of a star, local homogeneity is a good approximation. The relation between energy flux and the local temperature gradient derived with the assumption of $S_\nu = B_\nu$, which is good for large τ_* (including scattering in the source function), is called the **Rosseland approximation**.

Let's further take the plane-parallel assumption, that is, all physical properties of the medium depending on the depth only. The intensity is therefore only a function of depth z and polar direction θ. Considering that $\mathrm{d}s = \mathrm{d}z/\cos\theta$ and defining $\mu = \cos\theta$, we have

$$\mu \frac{\partial I_\nu(z,\mu)}{\partial z} = -(\alpha_\nu + \sigma_\nu)(I_\nu - S_\nu) \ , \tag{1.57}$$

and therefore

$$I_\nu(z,\mu) = S_\nu - \frac{\mu}{\alpha_\nu + \sigma_\nu} \frac{\partial I_\nu}{\partial z} \ . \tag{1.58}$$

Under the Rosseland approximation, $S_\nu = B_\nu$. The second term at the right hand side is of the order of B_ν/τ, which is very small compared to the first term because of a large τ. In such a case, we may take B_ν as the zeroth order approximation of I_ν and have

$$I_\nu(z,\mu) \approx B_\nu(T) - \frac{\mu}{\alpha_\nu + \sigma_\nu} \frac{\partial B_\nu}{\partial z} \ . \tag{1.59}$$

Sometimes the Rosseland approximation is referred to this equation.

The flux density in a certain direction is obtained by integrating the specific intensity over the solid angle in all directions, that is,

$$\begin{aligned} F_\nu(z) &= \int I_\nu \cos\theta \mathrm{d}\Omega \\ &= 2\pi \int_{-1}^{+1} I_\nu \mu \mathrm{d}\mu \\ &= \frac{-2\pi}{\alpha_\nu + \sigma_\nu} (\int_{-1}^{+1} \mu^2 \mathrm{d}\mu) \frac{\partial B_\nu}{\partial z} \\ &= -\frac{4\pi}{3(\alpha_\nu + \sigma_\nu)} \frac{\partial B_\nu}{\partial T} \frac{\partial T}{\partial z} \quad . \end{aligned} \tag{1.60}$$

The total flux is then

$$\begin{aligned} F(z) &= \int F_\nu(z) \mathrm{d}\nu \\ &= -\frac{4\pi}{3} \left(\int \frac{1}{(\alpha_\nu + \sigma_\nu)} \frac{\partial B_\nu}{\partial T} \mathrm{d}\nu \right) \frac{\partial T}{\partial z} \quad . \end{aligned} \tag{1.61}$$

Noting that $\int \frac{\partial B_\nu}{\partial T} \mathrm{d}\nu = \frac{4\sigma T^3}{\pi}$ and defining the Rosseland mean absorption coefficient α_{R} as

$$\frac{1}{\alpha_{\mathrm{R}}} = \frac{\int \frac{1}{(\alpha_\nu + \sigma_\nu)} \frac{\partial B_\nu}{\partial T} \mathrm{d}\nu}{\int \frac{\partial B_\nu}{\partial T} \mathrm{d}\nu} \quad , \tag{1.62}$$

we reach

$$F(z) = -\frac{16}{3} \frac{\sigma T^3}{\alpha_{\mathrm{R}}} \frac{\partial T}{\partial z} \quad . \tag{1.63}$$

Readers should not confuse the Stefan-Boltzmann constant σ with the scattering coefficient σ_ν. Eq.(1.63) is the Rosseland approximation of the energy flux. It is like an equation of heat conduction with a heat conductivity $\frac{16}{3} \frac{\sigma T^3}{\alpha_{\mathrm{R}}}$. It is also in a form of energy diffusion and this approximation is also called the **diffusion approximation**. A corresponding **Rosseland mean opacity** can be easily defined as $\kappa_{\mathrm{R}} = \alpha_{\mathrm{R}}/\rho$. Although an isotropic scattering is assumed in formulating Eq.(1.60), these same results can be obtained for the case of Thomson scattering, which has a forward-backward symmetry containing the square of the cosine of the scattering angle (Clayton 1983, p.181).

The Rosseland approximation assumes the specific intensity is close to the Planck function, which is valid for a large *effective* optical depth. Since thermal emission and scattering are both isotropic, intensity can become nearly isotropic at a large *ordinary* optical depth. It is therefore easier for the intensity to approach isotropy than to approach its thermal value. The assumption of near isotropy, which is called the **Eddington approximation**, is to consider the intensity up to the linear term in μ:

$$I(\tau, \mu) = a(\tau) + b(\tau)\mu \ . \tag{1.64}$$

The first three moments of this intensity is

$$J = \frac{1}{2}\int_{-1}^{+1} I \mathrm{d}\mu = a \ , \tag{1.65}$$

$$H = \frac{1}{2}\int_{-1}^{+1} \mu I \mathrm{d}\mu = \frac{b}{3} \ , \tag{1.66}$$

$$K = \frac{1}{2}\int_{-1}^{+1} \mu^2 I \mathrm{d}\mu = \frac{a}{3} \ . \tag{1.67}$$

J is the mean intensity and H and K are proportional to the flux and radiation pressure. We now have

$$K = \frac{1}{3}J \ , \tag{1.68}$$

which is also known as the Eddington approximation. We note that this is equivalent to $P_\nu = \frac{1}{3}u_\nu$ below Eq.(1.5), where an isotropic radiation field is assumed. Here we see that this result is valid also for a slightly anisotropic field containing a term linear in $\cos\theta$.

Defining the normal optical depth $\mathrm{d}\tau = -(\alpha_\nu + \sigma_\nu)\mathrm{d}z$, we have, from Eq.(1.57),

$$\mu\frac{\partial I}{\partial \tau} = I - S \ . \tag{1.69}$$

Directly integrating the above equation over μ, and multiplying a factor of μ before integrating, we get

$$\frac{\partial H}{\partial \tau} = J - S \ , \tag{1.70}$$

and

$$\frac{\partial K}{\partial \tau} = H \ . \tag{1.71}$$

The last two equations yield

$$\frac{1}{3}\frac{\partial^2 J}{\partial \tau^2} = J - S \ . \tag{1.72}$$

Using Eqs.(1.52) and (1.54), we then have

$$\frac{1}{3}\frac{\partial^2 J}{\partial \tau^2} = \epsilon(J - B) \ . \tag{1.73}$$

This equation is called the **radiative diffusion equation**, which is sometimes also used for Eq.(1.63). Given the physical structure of the medium, $B(\tau)$ and $\epsilon(\tau)$ are known. One may solve J, and then S, and then I.

If ϵ does not depend on depth, we may define a new optical depth

$$\mathrm{d}\tau_* := \sqrt{3\epsilon}\mathrm{d}\tau = \sqrt{3\alpha_\nu(\alpha_\nu + \sigma_\nu)}\mathrm{d}z \tag{1.74}$$

and have the transfer equation as

$$\frac{\partial^2 J}{\partial \tau_*^2} = J - B \ . \tag{1.75}$$

This equation shows the thermalization property of τ_*, similar to the effective optical depth discussed in the last section, but now more locally defined. One can see the thermalization from the solution - for simplicity assuming B is a constant - that $J = B + c_1 e^{-\tau_*} + c_2 e^{\tau_*}$ and c_2 is usually set to be zero for a finite value of J at $\tau_* \to \infty$.

Chapter 2

Polarization of Radiation Fields

2.1 Polarization and Stokes parameters: monochromatic waves

A monochromatic wave, at a location designated as $\vec{r} = 0$, can be described as the real part of the followng:

$$\vec{E} = (E_1\hat{x} + E_2\hat{y})e^{-i\omega t} \quad , \tag{2.1}$$

with

$$E_1 = \mathcal{E}_x e^{i\phi_x}, \qquad E_2 = \mathcal{E}_y e^{i\phi_y} \quad . \tag{2.2}$$

We will try to keep the convention that quantities with subscript 1 or 2 are complex and those with subscript x or y are real. Therefore we have

$$E_x = \mathcal{E}_x \cos(\omega t - \phi_x), \qquad E_y = \mathcal{E}_y \cos(\omega t - \phi_y) \quad . \tag{2.3}$$

This is in general elliptically polarized, depending on the phase difference $\phi_x - \phi_y$. It is linearly polarized when $\phi_x - \phi_y$ equals 0 or $\pm\pi$ and circularly polarized when $\phi_x - \phi_y$ equals $\pm\frac{\pi}{2}$.

One may also express this ellipse with respect to its principal axes, denoted as x'- and y'- axes, which are tilted at an angle χ to the original x- and y- axes, as shown in Figure 2.1. Then in general for an ellipse we have

$$E'_x = \mathcal{E}_0 \cos\beta \cos\omega t, \qquad E'_y = -\mathcal{E}_0 \sin\beta \sin\omega t \quad , \tag{2.4}$$

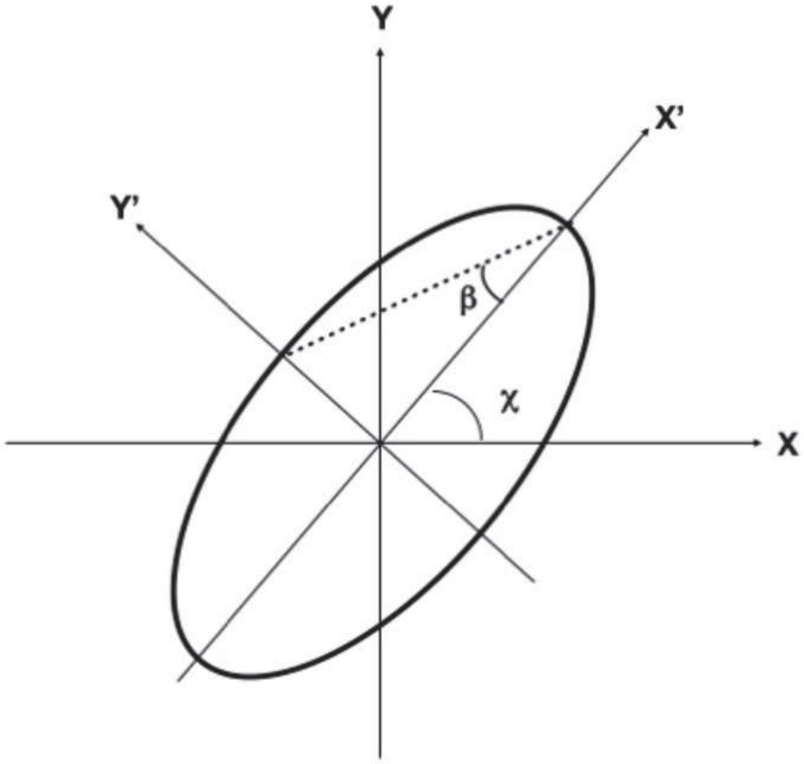

Figure 2.1: An ellipse traced out by the electric field of an electromagnetic wave at a certain location. Its propogation is in z direction, i.e., out of the page.

where $\mathcal{E}_0 \cos\beta$ and $\mathcal{E}_0 \sin\beta$ are the two principal axes. The angle β takes a valus in $-\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$. If $0 < \beta < \frac{\pi}{2}$, the electric field goes clockwise (note the minus sign in the prescription of E'_y) and it is called right-handed elliptical polarization or negative helicity. For $-\frac{\pi}{2} < \beta < 0$, it is left-handed and positive helicity. If $\beta = \pm\frac{\pi}{4}$, it is circularly polarized, while for $\beta = 0$ or $\pm\frac{\pi}{2}$, it has linear polarization in orthogonal directions.

One can link the above two expressions by noting that

$$E_x = E'_x \cos\chi - E'_y \sin\chi \;\; ; \;\; E_y = E'_x \sin\chi - E'_y \cos\chi$$

to get

$$E_x = \mathcal{E}_0(\cos\beta\cos\chi\cos\omega t + \sin\beta\sin\chi\sin\omega t)$$

and

$$E_y = \mathcal{E}_0(\cos\beta\sin\chi\cos\omega t - \sin\beta\cos\chi\sin\omega t) \ .$$

Comparing the above with Eq.(2.3), we have

$$\begin{aligned}
\mathcal{E}_x\cos\phi_x &= \mathcal{E}_0\cos\beta\cos\chi \ , && (2.5)\\
\mathcal{E}_x\sin\phi_x &= \mathcal{E}_0\sin\beta\sin\chi \ , && (2.6)\\
\mathcal{E}_y\cos\phi_y &= \mathcal{E}_0\cos\beta\sin\chi \ , && (2.7)\\
\mathcal{E}_y\sin\phi_y &= -\mathcal{E}_0\sin\beta\cos\chi \ . && (2.8)
\end{aligned}$$

We see that given $\mathcal{E}_x$, $\mathcal{E}_y$, ϕ_x and ϕ_y (in fact only $(\phi_x-\phi_y)$ matters), $\mathcal{E}_0$, β and χ can be determined, and vice versa. A convenient and conventional way of doing that is to define the **Stokes parameters** as the following:

$$\begin{aligned}
I &:= \mathcal{E}_x^2+\mathcal{E}_y^2 &&= \mathcal{E}_0^2 \ , && (2.9)\\
Q &:= \mathcal{E}_x^2-\mathcal{E}_y^2 &&= \mathcal{E}_0^2\cos 2\beta\cos 2\chi \ , && (2.10)\\
U &:= 2\mathcal{E}_x\mathcal{E}_y\cos(\phi_x-\phi_y) &&= \mathcal{E}_0^2\cos 2\beta\sin 2\chi \ , && (2.11)\\
V &:= 2\mathcal{E}_x\mathcal{E}_y\sin(\phi_x-\phi_y) &&= \mathcal{E}_0^2\sin 2\beta \ . && (2.12)
\end{aligned}$$

From the above definition we have

$$\begin{aligned}
\mathcal{E}_0 &= \sqrt{I} && (2.13)\\
\sin 2\beta &= \frac{V}{I} \ , && (2.14)\\
\tan 2\chi &= \frac{U}{Q} \ , && (2.15)
\end{aligned}$$

and

$$I^2 = Q^2+U^2+V^2 \ . \quad (2.16)$$

A monochromatic electromagnetic wave therefore can be characterized by $\{\mathcal{E}_x,\mathcal{E}_y,(\phi_x-\phi_y)\}$, $\{\mathcal{E}_0,\beta,\chi\}$, or $\{I,Q,U,V\}$. It is 100% elliptically polarized, with linear or circular polarization being two extreme cases. With the above definition, the Stokes parameter I is proportional to intensity or flux, and V is the circularity: a pure circular polarization when $|V|=I$ and a linear polarization when $V=0$. The sign of V indicates the sense of polarization (including elliptical one): right-handed when $V>0$ and left-handed when $V<0$. We also note that for $\chi\to\chi+\frac{\pi}{2}$ we have $\{Q,U\}\to\{-Q,-U\}$.

2.2 Polarization and Stokes parameters: quasi-monochromatic waves

A monochromatic electromagnetic wave is an idealization. In reality we may have $\mathcal{E}_x, \mathcal{E}_y$, and $(\phi_x - \phi_y)$ all being functions of time. If the variation is fast, polarization may not be well defined. One has an unpolarized light. If we consider a quasi-monochromatic wave, that is, a wave with a very small bandwidth $\triangle\omega$ ($\triangle\omega \ll \omega$), we may expect that within the coherence time $\triangle t$ ($\triangle t \triangle\omega \sim 1$), $\mathcal{E}_x, \mathcal{E}_y$, and $(\phi_x - \phi_y)$ vary with time only slowly and polarization states may still be identified to some extent. We note that in such a case $\triangle t \sim \frac{1}{\triangle\omega} \gg \frac{1}{\omega}$, that is, the coherence time is much longer than the period, because of a narrow bandwidth.

In real measurements, what is measured is usually the time-averaged square of the electric field strength. Very often the measuring devices form a linear combination of the two independent electric field components before feeding them into the detector. This linear combination can be generally written as $E'_i = \lambda_{ij} E_j$, i.e.,

$$\begin{aligned} E'_1 &= \lambda_{11} E_1 + \lambda_{12} E_2 \\ E'_2 &= \lambda_{21} E_1 + \lambda_{22} E_2 \ , \end{aligned} \tag{2.17}$$

where λ_{ij} is the device response.

Note that for $A = A_0 e^{i\omega t}$ and $B = B_0 e^{i\omega t}$, we have $\langle \mathrm{Re}\{A\} \times \mathrm{Re}\{B\} \rangle_t = \frac{1}{2}\mathrm{Re}\{A_0 B_0^*\}$, if the average is over a long enough time. We therefore have

$$\langle (\mathrm{Re}\{E'_1 e^{-i\omega t}\})^2 \rangle = \frac{1}{2} \langle \mathrm{Re}\{E'_1 \times E'^*_1\} \rangle \tag{2.18}$$

$$\begin{aligned} &= \frac{1}{2}(|\lambda_{11}|^2 \langle E_1 E_1^* \rangle + \lambda_{11}\lambda_{12}^* \langle E_1 E_2^* \rangle \\ &+ \lambda_{11}^* \lambda_{12} \langle E_1^* E_2 \rangle + |\lambda_{12}|^2 \langle E_2 E_2^* \rangle) \ . \end{aligned} \tag{2.19}$$

The average over time at the left-hand side takes care of the fast variation in $e^{-i\omega t}$, while that at the right-hand side refers only to the average over the slow variation of E_1 and E_2 during the time interval of the measurement.

We see that four quantities, $\langle E_i E_j^* \rangle$, determine the measurement. They are in fact composed of four real numbers, which can be measured. One usually uses Stokes parameters for quasi-monochromatic waves to express these $\langle E_i E_j^* \rangle$ as the following:

$$
\begin{aligned}
I &:= \quad \langle E_1 E_1^* \rangle + \langle E_2 E_2^* \rangle &&= \langle \mathcal{E}_x^2 + \mathcal{E}_y^2 \rangle \ , && (2.20)\\
Q &:= \quad \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle &&= \langle \mathcal{E}_x^2 - \mathcal{E}_y^2 \rangle \ , && (2.21)\\
U &:= \quad \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle &&= \langle 2\mathcal{E}_x \mathcal{E}_y \cos(\phi_x - \phi_y) \rangle \ , && (2.22)\\
V &:= \ \tfrac{1}{i} \left(\langle E_1 E_2^* \rangle - \langle E_2 E_1^* \rangle \right) &&= \langle 2\mathcal{E}_x \mathcal{E}_y \sin(\phi_x - \phi_y) \rangle \ . && (2.23)
\end{aligned}
$$

From the **Schwarz inequality**, which can be derived by considering $\int |f + \lambda g|^2 \mathrm{d}t \geq 0$ for any arbitrary complex functions f and g and the choice of $\lambda = -\int g^* f \mathrm{d}t / \int g^* g \mathrm{d}t$,

$$
\langle E_1 E_1^* \rangle \langle E_2 E_2^* \rangle \geq \langle E_1 E_2^* \rangle \langle E_2 E_1^* \rangle \ , \tag{2.24}
$$

we have

$$
I^2 \geq Q^2 + U^2 + V^2 \ . \tag{2.25}
$$

The equality holds when the ratio E_1/E_2 is constant in time. This will be a wave with a 100% polarization, similar to the case for a monochromatic wave. If E_1 and E_2 are totally unrelated and there is no preferred direction in the X-Y plane, we will have $Q = U = V = 0$, i.e., a completely unpolarized light.

To describe a partially polarized wave, we note that Stokes parameters are *additive* for *independent* waves. By 'independent' we mean there is no permanent relation between phases of the waves and the relative phases can be assumed to be randomly and uniformly distributed in $\{0, 2\pi\}$. To see this, consider that

$$
E_1 = \sum_k E_1^{(k)}, \ \ E_2 = \sum_\ell E_2^{(\ell)} \tag{2.26}
$$

and

$$
\begin{aligned}
\langle E_i E_j^* \rangle &= \sum_k \sum_\ell \langle E_i^{(k)} E_j^{(\ell)*} \rangle && (2.27)\\
&= \sum_k \langle E_i^{(k)} E_j^{(k)*} \rangle \ , && (2.28)
\end{aligned}
$$

where the last equality comes from the fact that phases are random. We therefore have

$$\begin{aligned}
I &= \sum_k I^{(k)} \\
Q &= \sum_k Q^{(k)} \\
U &= \sum_k U^{(k)} \\
V &= \sum_k V^{(k)} \ .
\end{aligned} \tag{2.29}$$

Therefore, Stokes parameters of a wave can be generally decomposed into

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} \sqrt{Q^2+U^2+V^2} \\ Q \\ U \\ V \end{pmatrix} + \begin{pmatrix} I-\sqrt{Q^2+U^2+V^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \ . \tag{2.30}$$

It is a completely polarized wave plus an unpolarized wave. Such a wave is called partially polarized. Its **degree of polarization** is defined as

$$\Pi = \frac{I_{\rm pol}}{I} = \frac{\sqrt{Q^2+U^2+V^2}}{I} \ . \tag{2.31}$$

For $V=0$, i.e., a partially linear polarization, $I_{\rm max}$ and $I_{\rm min}$ in certain two perpendicular directions can be measured and its polarization degree is

$$\Pi = \frac{I_{\rm max}-I_{\rm min}}{I_{\rm max}+I_{\rm min}} \ , \tag{2.32}$$

because $I_{\rm max} = \frac{1}{2}I_{\rm unpol} + I_{\rm pol}$ and $I_{\rm min} = \frac{1}{2}I_{\rm unpol}$. This can be seen by noting that an unpolarized wave can be decomposed into two waves completely linearly polarized in perpendicular directions as

$$\begin{pmatrix} I_{\rm unpol} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I_{\rm unpol} \\ q \\ u \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}I_{\rm unpol} \\ -q \\ -u \\ 0 \end{pmatrix} \ , \tag{2.33}$$

where $(\frac{1}{2}I_{\rm unpol})^2 = q^2+u^2$ and $u/q = U/Q$. Eq.(2.32) is for linear polarization only. It underestimates the degree of polarization if the polarization is not linear.

Chapter 3

Radiation from Non-Relativistic Charges

3.1 Larmor's formula

For non-relativistic charges, i.e., $\beta \ll 1$, the radiation fields are

$$\vec{E}_{\rm rad} = \frac{q}{Rc^2}(\hat{n} \times (\hat{n} \times \dot{\vec{u}})) \tag{3.1}$$

$$\vec{B}_{\rm rad} = \hat{n} \times \vec{E}_{\rm rad} \ , \tag{3.2}$$

where q is the electric charge of the charged particles, R and $\hat{n}$ are the distance and the unit vector from the charge to the field point, and $\vec{u}$ is the velocity of the charge. All the quantities at the right-hand side are evaluated at the retarded time. We note that $\vec{E}_{\rm rad}$ lies in the plane spanned by $\dot{\vec{u}}$ and $\hat{n}$. The magnitude of these fields is

$$|\vec{E}_{\rm rad}| = |\vec{B}_{\rm rad}| = \frac{q\dot{u}}{Rc^2}\sin\theta \ , \tag{3.3}$$

where θ is the angle between $\dot{\vec{u}}$ and $\hat{n}$. We therefore have the magnitude of the Poynting vector as

$$S = \frac{c}{4\pi}E^2 = \frac{c}{4\pi}\frac{q^2\dot{u}^2}{R^2c^4}\sin^2\theta \ . \tag{3.4}$$

The Poynting vector is an energy flux, that is, $S = \frac{\mathrm{d}W}{\mathrm{d}t\mathrm{d}A} = \frac{\mathrm{d}W}{\mathrm{d}t\mathrm{d}\Omega R^2}$. We therefore have the power per solid angle in a certain direction as

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{q^2\dot{u}^2}{4\pi c^3}\sin^2\theta \ . \tag{3.5}$$

The radiation has a $\sin^2\theta$ dependence and is strongest in the direction perpendicular to the acceleration $\dot{\vec{u}}$. The total powwer is

$$P = \int \frac{\mathrm{d}P}{\mathrm{d}\Omega}\mathrm{d}\Omega = \frac{2q^2\dot{u}^2}{3c^3} \ . \tag{3.6}$$

This is called **Larmor's formula**, which says the total radiation power of a non-relativistic charge is proportional to the product of charge squared and acceleration squared, i.e., $q^2\dot{u}^2$. In the following sections we will apply Larmor's formula to dipole radiation, Thomson scattering and radiation from harmonically bound charges.

3.2 Dipole radiation

In the previous section, one single non-relativistic charge is treated. When we have a system of charges, their radiation fields need to be added at different retarded times. A certain simplification, however, can be justified for the non-relativistic case. Let L be the system size and τ be the time scale of change in the system configuration. The differences in retarded times can be ignored if $\tau \gg L/c$. Consider ℓ being the scale of particles' orbits. The above condition is the same as $\ell/u \gg L/c$, that is, $u/c \ll \ell/L < 1$, the non-relativistic condition.

In such an approximation, the radiation fields of each charge can be added together and evaluated at the same time. The field at a large distance can be written as

$$\begin{aligned}\vec{E} &= \sum_i \left(\frac{q_i}{R_i c^2}(\hat{n}_i \times (\hat{n}_i \times \dot{\vec{u}}_i))\right) \\ &= \frac{\hat{n} \times (\hat{n} \times \ddot{\vec{d}})}{Rc^2} \ ,\end{aligned} \tag{3.7}$$

where $\vec{d}$ is the dipole moment:

$$\vec{d} = \sum_i q_i \vec{r}_i \tag{3.8}$$

and R and $\hat{n}$ are the distance and the unit vector from the system of charges to the field point. From Eq.(3.5), we have

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{|\ddot{\vec{d}}|^2}{4\pi c^3}\sin^2\theta \ , \tag{3.9}$$

and

$$P = \frac{2|\ddot{\vec{d}}|^2}{3c^3} \ . \tag{3.10}$$

This is the total power of dipole radiation.

In general we have

$$E(t) = \frac{\ddot{d}}{Rc^2}\sin\theta \ . \tag{3.11}$$

To find the spectrum of dipole radiation, let's consider $\ddot{\vec{d}} \parallel \dot{\vec{d}} \parallel \vec{d}$ for simplicity. The dipole moment can be written in terms of its Fourier transform as

$$d(t) = \int \tilde{d}(\omega)e^{-i\omega t}\mathrm{d}\omega \ . \tag{3.12}$$

Then we have

$$\ddot{d(t)} = -\int \omega^2\tilde{d}(\omega)e^{-i\omega t}\mathrm{d}\omega \ , \tag{3.13}$$

and

$$\tilde{E}(\omega) = -\frac{\omega^2\tilde{d}(\omega)}{Rc^2}\sin\theta \ . \tag{3.14}$$

From $\mathrm{d}W/\mathrm{d}A = \frac{c}{4\pi}\int_{-\infty}^{\infty} E^2(t)\mathrm{d}t$ and $\int_{-\infty}^{\infty} E^2(t)\mathrm{d}t = 2\pi\int_{-\infty}^{\infty}|\tilde{E}(\omega)|^2\mathrm{d}\omega$, and then $\mathrm{d}W/\mathrm{d}A = c\int_0^{\infty}|\tilde{E}(\omega)|^2\mathrm{d}\omega$, we have $\frac{\mathrm{d}W}{\mathrm{d}\omega\mathrm{d}A} = c|\tilde{E}(\omega)|^2$. The differential spectrum in a certain direction is therefore

$$\frac{\mathrm{d}W}{\mathrm{d}\omega\mathrm{d}\Omega} = R^2c|\tilde{E}(\omega)|^2 \tag{3.15}$$

$$= \frac{\omega^4|\tilde{d}(\omega)|^2}{c^3}\sin^2\theta \ , \tag{3.16}$$

and the spectrum as

$$\frac{\mathrm{d}W}{\mathrm{d}\omega} = \frac{8\pi\omega^4}{3c^3}|\tilde{d}(\omega)|^2 \ . \tag{3.17}$$

The dipole radiation spectrum is therefore directly related to the frequency of dipole oscillation, that is, the radiated emission is at the frequency of the dipole oscillation. It is, however, not the case for ultra-relativistic charges, which we will discuss in the next chapter.

3.3 Thomson scattering

The scattering of low energy ($h\nu \ll m_\mathrm{e}c^2$) photons off electrons at rest can be treated with classical electrodynamics. Photons do not change energy after scattering, as one can see from the dipole radiation spectrum as long as electrons stay non-relativistic. This is called Thomson scattering. To find the scattering cross section, let's start with Larmor's formula.

Consider for the moment a *linearly polarized* incoming light with electric field $\vec{E}_\mathrm{in}$. We then have $\dot{\vec{u}} = \frac{q}{m}\vec{E}_\mathrm{in}$ and $S_\mathrm{in} = \frac{c}{4\pi}|\vec{E}_\mathrm{in}|^2$. Now the differential power, Eq.(3.5), becomes, taking $q = -e$ and $m = m_\mathrm{e}$,

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{e^4}{m_\mathrm{e}^2c^4}S_\mathrm{in}\sin^2\theta \ . \tag{3.18}$$

The differential scattering cross section is $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}P}{\mathrm{d}\Omega}/S_\mathrm{in}$, that is,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{e^2}{m_\mathrm{e}c^2}\right)^2 \sin^2\theta = r_\mathrm{e}^2\sin^2\theta \ , \tag{3.19}$$

where r_e is the classical radius of electrons. This is the differential Thomson scattering cross section. To get the total cross section we integrate the differential one over all solid angle. The result is $\sigma_\mathrm{T} = \frac{8\pi}{3}r_\mathrm{e}^2$.

Now consider an *unpolarized* incoming light. We may decompose this light into two components of mutually perpendicular, linearly polarized lights. We may further choose $\hat{e}_1$ in the plane of $\hat{n}$ and $\hat{k}$, where $\hat{e}_1$ is the direction of the polarization of one incoming light component and $\hat{k}$ is the propagation direction, and choose $\hat{e}_2 = \hat{k} \times \hat{e}_1$; see Figure 3.1. Therefore the differential

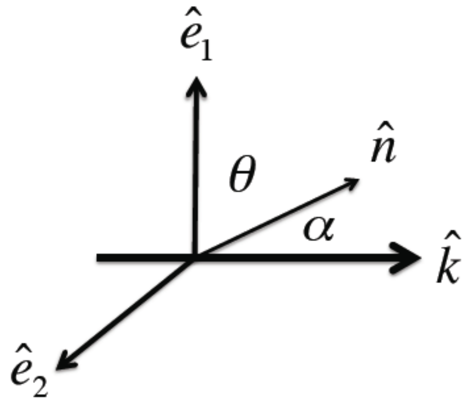

Figure 3.1: $\hat{n}$ is in the plane of $\hat{e}_1$ and $\hat{k}$.

cross section is

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\theta) + \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\frac{\pi}{2})\right) && (3.20)\\
&= \frac{r_\mathrm{e}^2}{2}(\cos^2\alpha + 1)\ , && (3.21)
\end{aligned}$$

where the right-hand side is taken from the polarized cross section and α is the scattering angle, i.e., the angle between incoming and outgoing light, $\cos\alpha = \hat{n}\cdot\hat{k}$, as shown in Figure 3.1. The total cross section is then

$$\begin{aligned}
\sigma_\mathrm{T} &= 2\pi\int\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}(\cos\alpha)\\
&= \pi r_\mathrm{e}^2\int_{-1}^{1}(\cos^2\alpha + 1)\mathrm{d}(\cos\alpha)\\
&= \frac{8\pi}{3}r_\mathrm{e}^2\ . && (3.22)
\end{aligned}$$

From the above discussion, we see that the total cross section for Thomson scattering, no matter incoming light being polarized or not, is

$$\sigma_\mathrm{T} = \frac{8\pi}{3}r_\mathrm{e}^2 = 0.665\times10^{-24}\mathrm{cm}^2\ . \qquad (3.23)$$

This cross section is called the **Thomson cross section**. It is very often used to estimate the magnitude of interaction between light and matter.

A polarized light after Thomson scattering still keeps its polarization state. For unpolarized incoming light, the scattered light is in general partially linearly polarized. Recall the $\sin^2\theta$ dependence and consider contributions from the two decomposed components. We can see the polarization degree will be (Eq.(2.32))

$$\Pi = \frac{1-\cos^2\alpha}{1+\cos^2\alpha} \ . \tag{3.24}$$

The polarization is in the direction perpendicular to the plane of $\hat{n}$ and $\hat{k}$. The polarization degree Π approaches 100% when α approaches 90°, i.e., perpendicular scattering.

3.4 Radiation from harmonically bound systems

A charge that is bound to a center by a force like $\vec{F} = -k\vec{r} = -m\omega_0^2\vec{r}$ will oscillate sinusoidally with frequency ω_0. Such a charge will radiate like a varying dipole. This model is a highly idealized one but it provides a classical model for a spectral line. Damping is present because of the loss of radiated energy. Let's assume the radiation reaction force can be treated as perturbation. This condition is to require the time scale for energy change be much larger than that for particle orbital motion, that is,

$$\frac{mu^2}{P} \gg \frac{u}{\dot{u}} \ . \tag{3.25}$$

From $P = \frac{2q^2\dot{u}^2}{3mc^3}$, the above condition leads to $\frac{u}{\dot{u}} \gg \tau$, where $\tau = \frac{2q^2}{3mc^3}$. For electrons, $\tau \sim 10^{-23}$s. In such a case, $\omega_0\tau \ll 1$ and the reaction force is $\vec{F} = m\tau\ddot{\vec{u}}$ (Rybicki & Lightman (1979), Section 3.5).

For simplicity, let's consider the charge motion is one dimensional. The equation of motion reads

$$\ddot{x} = -\omega_0 x + \tau\ddot{u} \ . \tag{3.26}$$

Since the damping is weak, we may approxmate $\ddot{u}$ in terms of $\dot{x}$ by taking $x(t) \propto \cos(\omega_0 t + \phi)$ as

$$\ddot{u} \approx -\omega_0^2\dot{x} \ . \tag{3.27}$$

Then, the equation of motion turns into that for a damped oscillation:

$$\ddot{x} + \omega_0^2 \tau \dot{x} + \omega_0^2 x = 0 \ . \tag{3.28}$$

The solution to this equation is (more precisely speaking, when $\frac{\Gamma}{2} < \omega_0$)

$$x(t) = x_0 e^{-\Gamma t/2} \cos(\omega' t + \phi) \ , \tag{3.29}$$

where $\Gamma = \omega_0^2 \tau$ and $\omega' = \sqrt{\omega_0^2 - (\Gamma/2)^2}$. Since $\Gamma \ll \omega_0$ (because $\omega_0 \tau \ll 1$) and with a certain initial condition, we may have

$$x(t) = x_0 e^{-\Gamma t/2} \cos \omega_0 t \ . \tag{3.30}$$

The Fourier transform of $x(t)$ is

$$\tilde{x}(\omega) = \frac{1}{2\pi} \int_0^\infty x(t) e^{i\omega t} \mathrm{d}t \tag{3.31}$$

$$= \frac{x_0}{4\pi} \left(\frac{1}{\Gamma/2 - i(\omega + \omega_0)} + \frac{1}{\Gamma/2 - i(\omega - \omega_0)} \right) \ . \tag{3.32}$$

Note that we consider $t > 0$ only. Since we are interested only in positive frequencies and in regions of large values, we may take the following approximation:

$$\tilde{x}(\omega) \approx \frac{x_0}{4\pi} \frac{1}{\Gamma/2 - i(\omega - \omega_0)} \ , \tag{3.33}$$

and

$$|\tilde{x}(\omega)|^2 = \left(\frac{x_0}{4\pi} \right)^2 \frac{1}{(\omega - \omega_0)^2 + (\Gamma/2)^2} \ . \tag{3.34}$$

From Eq.(3.17), we have the spectrum as

$$\frac{\mathrm{d}W}{\mathrm{d}\omega} = \frac{8\pi \omega^4 q^2}{3c^3} \left(\frac{x_0}{4\pi} \right)^2 \frac{1}{(\omega - \omega_0)^2 + (\Gamma/2)^2} \ . \tag{3.35}$$

One can see from the above that Γ is the full width at half maximum (FWHM). One can also put the above equation as

$$\frac{\mathrm{d}W}{\mathrm{d}\omega} = \left(\frac{1}{2} m \omega_0^2 x_0^2\right) \left(\frac{\Gamma/2\pi}{(\omega - \omega_0)^2 + (\Gamma/2)^2} \right) \ . \tag{3.36}$$

The first factor is the initial potential energy of the system and the second factor is a **Lorentzian profile**, which satisfies the following normalization condition:

$$\int_{-\infty}^{\infty} \frac{\Gamma/2\pi}{(\omega-\omega_0)^2+(\Gamma/2)^2} \mathrm{d}\omega = \frac{1}{\pi}\tan^{-1}\left(\frac{2(\omega-\omega_0)}{\Gamma}\right)\Big|_{-\infty}^{\infty} = 1 \ . \qquad (3.37)$$

A correct description of atomic or molecular line emission requires quantum mechanics. What is learned in this section is that, through a classical toy model, one sees that the spectrum of a radiating system with a finite life time has a Lorentzian profile. The life time $\triangle T$, taken as $\triangle T = \int_0^\infty t|x(t)|^2 \mathrm{d}t / \int_0^\infty |x(t)|^2 \mathrm{d}t$, is $\triangle T = 1/\Gamma$. We therefore have the product of life time and the spectrum FWHM being unity, i.e., $\triangle T\Gamma = 1$.

Chapter 4

Radiation from Relativistic Charges

4.1 The Lorentz Transformation, Beaming, and Doppler effect

The space-time coordinates of an event in two frames, K and K', with K' moving at speed v towards the $+X$-axis direction, are related by the Lorentz transformation:

$$x' = \gamma(x - vt) \tag{4.1}$$

$$y' = y \tag{4.2}$$

$$z' = z \tag{4.3}$$

$$t' = \gamma(t - \frac{v}{c^2}x) \ , \tag{4.4}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \ , \ \beta = \frac{v}{c} \ . \tag{4.5}$$

If taking the space-time position four-vector to be $x^0 = ct, x^1 = x, x^2 = y$, and $x^3 = z$, the Lorentz transformation can be written as

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \tag{4.6}$$

and we have

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \quad . \tag{4.7}$$

The velocity $\vec{u}$ of a particle in frame K is related to its velocity $\vec{u}'$ in frame K' as

$$\begin{aligned}
u_x &= \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{u'_x + v}{1 + \beta u'_x/c} &(4.8)\\
u_y &= \frac{u'_y}{\gamma(1 + \beta u'_x/c)} &(4.9)\\
u_z &= \frac{u'_z}{\gamma(1 + \beta u'_x/c)} \quad . &(4.10)
\end{aligned}$$

It can also be generally written as

$$\begin{aligned}
u_{\parallel} &= \frac{u'_{\parallel} + v}{1 + \beta u'_{\parallel}/c} &(4.11)\\
u_{\perp} &= \frac{u'_{\perp}}{\gamma(1 + \beta u'_{\parallel}/c)} \quad . &(4.12)
\end{aligned}$$

From this we may have the **aberration formula** for the directions of the two velocities:

$$\tan\theta = \frac{u_{\perp}}{u_{\parallel}} = \frac{u'_{\perp}}{\gamma(u'_{\parallel} + v)} = \frac{u' \sin\theta'}{\gamma(u' \cos\theta' + v)} \quad . \tag{4.13}$$

The azimuthal angle is the same in the two frames, i.e., $\phi = \phi'$.

If we consider the case of $u' = c$, we will have

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)} \quad , \tag{4.14}$$

and

$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'} \quad . \tag{4.15}$$

These are the aberration of light. If we further consider the case of $\theta' = \frac{\pi}{2}$, we have

$$\sin\theta = \frac{1}{\gamma} \quad . \tag{4.16}$$

For $\gamma \gg 1$, we have $\theta \sim \frac{1}{\gamma}$. It means that all the emission from a relativistic particle in the forward half hemisphere will be beamed into a forward narrow beam of half angle $\frac{1}{\gamma}$. This is called the **relativistic beaming effect**.

Now let's consider the **Doppler effect** of electromagnetic waves emitted by a moving source. The time interval for a particle to emit one period of radiation in its comoving frame K' is $2\pi/\omega'$. That time interval, as measured in the frame K, will be

$$\triangle t_{\rm e} = \gamma \frac{2\pi}{\omega'} \tag{4.17}$$

because of time dilation. The time difference of the arrival of the start and the end of that period of radiation is then

$$\triangle t_{\rm a} = \triangle t_{\rm e} - \frac{d}{c} = \triangle t_{\rm e}(1 - \frac{v\cos\theta}{c}) \ , \tag{4.18}$$

where $d = \triangle t_{\rm e} v \cos\theta$ is the distance that the source travels when emitting that period of radiation. Here v is the speed of the particle, which always takes a positive value, and θ is the angle between particle motion and emission towards the observer. The observed frequency ω is then

$$\omega = \frac{2\pi}{\triangle t_{\rm a}} = \frac{\omega'}{\gamma(1-\beta\cos\theta)} \ . \tag{4.19}$$

It can also be written as

$$\omega' = \omega\gamma(1-\beta\cos\theta) \tag{4.20}$$

and, from Eq.(4.15),

$$\omega = \omega'\gamma(1+\beta\cos\theta') \ . \tag{4.21}$$

Another way of reaching Eq.(4.20) is to consider that the energy-momentum four-vector of a photon is

$$k^\mu = \begin{pmatrix} \omega/c \\ \vec{k} \end{pmatrix} \ , \tag{4.22}$$

and $|\vec{k}| = \omega/c$. Because $k^1 = (\omega/c)\cos\theta$ and

$$k'^0 = \gamma(k^0 - \beta k^1) \ , \tag{4.23}$$

we reach again Eq.(4.20).

4.2 Emission power of a relativistic particle

Suppose in the frame K', a total amount of energy $\mathrm{d}W'$ is emitted in $\mathrm{d}t'$ by a particle. The particle is at rest in K' instantaneously, so its (non-relativistic) emission is symmetric in any direction and its opposite direction (recall the $\sin^2\theta'$ dependence). The total momentum of this emitted radiation, $\mathrm{d}\vec{p}\,'$, is therefore zero. So, we have the total amount of energy $\mathrm{d}W$ in frame K to be $\mathrm{d}W = \gamma \mathrm{d}W'$ from the Lorentz transformation. We also have $\mathrm{d}t = \gamma \mathrm{d}t'$. The *total* emitted power in frames K and K' is $P = \mathrm{d}W/\mathrm{d}t$ and $P' = \mathrm{d}W'/\mathrm{d}t'$, so we have

$$P = P' \ . \tag{4.24}$$

The total power is a Lorentz invariant for a particle emitting with a front-back symmetry in its instantaneous rest frame.

Similar to the way to getting to Eq.(4.12), we may derive the relation between the three-vector acceleration in frames K and K' as the following:

$$a'_\parallel = \gamma^3 a_\parallel \tag{4.25}$$

$$a'_\perp = \gamma^2 a_\perp \ . \tag{4.26}$$

Here again K' is the instantaneous rest frame. From the invariance of the total power, we then have the relativistic version of Larmor's formula:

$$\begin{aligned} P &= \frac{2q^2}{3c^3}\vec{a}\,'\cdot\vec{a}\,' \\ &= \frac{2q^2}{3c^3}(a'^2_\parallel + a'^2_\perp) \\ &= \frac{2q^2}{3c^3}\gamma^4(\gamma^2 a^2_\parallel + a^2_\perp) \ . \end{aligned} \tag{4.27}$$

For the angular distribution of power, let's consider an amount of energy $\mathrm{d}W$ emitted into the solid angle $\mathrm{d}\Omega = \mathrm{d}\cos\theta\mathrm{d}\phi$ in in the direction at angle θ to the X-axis. Denoting $\mu = \cos\theta$ and $\mu' = \cos\theta'$ and from the transformation of the energy-momentum four-vector, we have

$$\mathrm{d}W = \gamma(\mathrm{d}W' + v\mathrm{d}p'_x) = \gamma(1+\beta\mu')\mathrm{d}W' \ . \tag{4.28}$$

From Eq.(4.15),

$$\mu = \frac{\mu' + \beta}{1+\beta\mu'} \ , \tag{4.29}$$

we have

$$\mathrm{d}\mu = \frac{\mathrm{d}\mu'}{\gamma^2(1+\beta\mu')^2} \ .$$

Since $\mathrm{d}\phi = \mathrm{d}\phi'$, the differential solid angles are

$$\mathrm{d}\Omega = \frac{\mathrm{d}\Omega'}{\gamma^2(1+\beta\mu')^2} \ ,$$

and we finally have the angular distribution of the emitted energy in the two frames to be related as

$$\frac{\mathrm{d}W}{\mathrm{d}\Omega} = \gamma^3(1+\beta\mu')^3\frac{\mathrm{d}W'}{\mathrm{d}\Omega'} \ . \tag{4.30}$$

The differential power emitted in frame K' is simply $\mathrm{d}P'/\mathrm{d}\Omega' = \mathrm{d}W'/\mathrm{d}t'\mathrm{d}\Omega'$, but in frame K we have two options:

1. The *emitted* power in frame K

$$\frac{\mathrm{d}P_\mathrm{e}}{\mathrm{d}\Omega} = \frac{\mathrm{d}W}{\mathrm{d}t_\mathrm{e}\mathrm{d}\Omega} = \frac{\mathrm{d}W}{\gamma\mathrm{d}t'\mathrm{d}\Omega} \tag{4.31}$$

2. The *received* power in frame K

$$\frac{\mathrm{d}P_\mathrm{r}}{\mathrm{d}\Omega} = \frac{\mathrm{d}W}{\mathrm{d}t_\mathrm{a}\mathrm{d}\Omega} = \frac{\mathrm{d}W}{\gamma(1-\beta\mu)\mathrm{d}t'\mathrm{d}\Omega} \tag{4.32}$$

The differential power is therefore

$$\frac{\mathrm{d}P_\mathrm{e}}{\mathrm{d}\Omega} = \gamma^2(1+\beta\mu')^3\frac{\mathrm{d}P'}{\mathrm{d}\Omega'} = \frac{1}{\gamma^4(1-\beta\mu)^3}\frac{\mathrm{d}P'}{\mathrm{d}\Omega'} \tag{4.33}$$

and

$$\frac{\mathrm{d}P_\mathrm{r}}{\mathrm{d}\Omega} = \gamma^4(1+\beta\mu')^4\frac{\mathrm{d}P'}{\mathrm{d}\Omega'} = \frac{1}{\gamma^4(1-\beta\mu)^4}\frac{\mathrm{d}P'}{\mathrm{d}\Omega'} \ , \tag{4.34}$$

where the relation $\gamma(1-\beta\mu) = \frac{1}{\gamma(1+\beta\mu')}$ as in Eq.(4.20) and Eq.(4.21) is used for the last equality in the above two equations.

Let's consider the received power for the moment. If the particle is highly relativistic, i.e., $\beta \sim 1$, the emission will be concentrated in the forward direction. Taking

$$\mu = \cos\theta \approx 1 - \frac{\theta^2}{2} \ , \tag{4.35}$$

and

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} \approx 1 - \frac{1}{2\gamma^2} \ , \tag{4.36}$$

it follows that

$$\frac{1}{\gamma^4(1-\beta\mu)^4} \approx \left(\frac{2\gamma}{1+\gamma^2\theta^2}\right)^4 \ . \tag{4.37}$$

It is indeed sharply peaked at $\theta \approx 0$ in the range of order $1/\gamma$, as we had earlier.

To visualize the angular distribution, we consider two special cases, that is, acceleration parallel and perpendicular to velocity respectively. From Eq.(3.5), the differential power in K' is

$$\frac{\mathrm{d}P'}{\mathrm{d}\Omega'} = \frac{q^2 a'^2}{4\pi c^3} \sin^2\Theta' \ , \tag{4.38}$$

where Θ' is the angle between the acceleration and the emission in K'. From Eq.(4.34) and Eq.(4.27), we have

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{q^2}{4\pi c^3} \frac{(\gamma^2 a_\parallel^2 + a_\perp^2)}{(1-\beta\mu)^4} \sin^2\Theta' \ , \tag{4.39}$$

We then discuss the two special cases in the following:
For the case of acceleration parallel to velocity, $a_\perp = 0 = a'_\perp$. We also have $\Theta' = \theta'$, where θ' is the angle between the velocity and emission in frame K'. From Eq.(4.15), or equivalently, $\mu' = (\mu - \beta)/(1 - \beta\mu)$, we have

$$\sin^2\Theta' = \frac{\sin^2\theta}{\gamma^2(1-\beta\mu)^2} \tag{4.40}$$

and therefore

$$\frac{\mathrm{d}P_\parallel}{\mathrm{d}\Omega} = \frac{q^2}{4\pi c^3} a_\parallel^2 \frac{\sin^2\theta}{(1-\beta\mu)^6} \ . \tag{4.41}$$

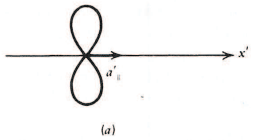

Figure 4.11a Dipole radiation pattern for particle at rest.

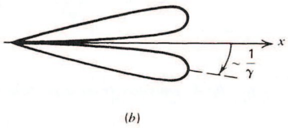

Figure 4.11b Angular distribution of radiation emitted by a particle with parallel acceleration and velocity.

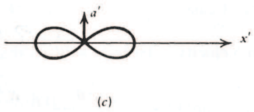

Figure 4.11c Same as a.

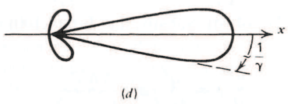

Figure 4.11d Angular distribution of radiation emitted by a particle with perpendicular acceleration and velocity.

Figure 4.1: Angular distribution of radiation from a relativistic particle in its rest frame K' and the observer's frame K. Taken from Rybicki & Lightman (1979), page 144.

For the extreme relativistic case, from Eq.(4.37), we see that

$$(1-\beta\mu) \approx \frac{1+\gamma^2\theta^2}{2\gamma^2} \tag{4.42}$$

and

$$\frac{\mathrm{d}P_{\parallel}}{\mathrm{d}\Omega} \approx \frac{16q^2}{\pi c^3} a_{\parallel}^2 \gamma^{10} \frac{\gamma^2\theta^2}{(1+\gamma^2\theta^2)^6} \quad . \tag{4.43}$$

For the case of acceleration perpendicular to velocity, $a_{\parallel} = 0 = a'_{\parallel}$. In such a case, there is no axial symmetry about the direction of particle's motion (the polar direction) and we have $\cos\Theta' = \sin\theta'\cos\phi'$, where ϕ' is the azimuthal angle with $\phi' = 0$ in the direction of the perpendicular acceleration. We therefore have

$$\sin^2\Theta' = 1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\mu)^2} \tag{4.44}$$

and

$$\frac{\mathrm{d}P_\perp}{\mathrm{d}\Omega} = \frac{q^2}{4\pi c^3}a_\perp^2\frac{1}{(1-\beta\mu)^4}\left(1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\mu)^2}\right) \ . \tag{4.45}$$

For the extreme relativistic case, it becomes

$$\frac{\mathrm{d}P_\perp}{\mathrm{d}\Omega} \approx \frac{4q^2}{\pi c^3}a_\perp^2\gamma^8\frac{1-2\gamma^2\theta^2\cos 2\phi + \gamma^4\theta^4}{(1+\gamma^2\theta^2)^6} \ . \tag{4.46}$$

We note that in both extreme relativistic cases, the θ dependence only appears in the form of $\gamma\theta$. Emissions are mainly within the half angle of $1/\gamma$, as shown in Figure 4.1, where Eq.(4.41) and Eq.(4.45) are plotted with a large γ.

4.3 Some Lorentz invariant quantities

The four-volume element d^4x is Lorentz invariant, i.e., $\mathrm{d}^4x' = \mathrm{d}^4x$, because the determinant of the Lorentz transformation matrix, the Jacobian, is unity. Here we discuss three more sets of invariant quantities:

- The phase-space element. Consider a group of particles occupying a phase-space element $\mathrm{d}^3\vec{x}\,'\mathrm{d}^3\vec{p}\,' = \mathrm{d}x_1'\mathrm{d}x_2'\mathrm{d}x_3'\mathrm{d}p_1'\mathrm{d}p_2'\mathrm{d}p_3'$ in their rest frame. Since $p_0' \propto p'^2$ (non-relativistic) and $\mathrm{d}p_0' \propto p'\mathrm{d}p'$, they have no spread in energy, i.e., $\mathrm{d}p_0' = 0$ (because $p' \approx 0$). Considering the length contraction, we have $\mathrm{d}x_1 = \gamma^{-1}\mathrm{d}x_1'$, $\mathrm{d}x_2 = \mathrm{d}x_2'$, and $\mathrm{d}x_3 = \mathrm{d}x_3'$. For the momentum, we have $\mathrm{d}p_1 = \gamma(\mathrm{d}p_1' + \beta\mathrm{d}p_0') = \gamma\mathrm{d}p_1'$, $\mathrm{d}p_2 = \mathrm{d}p_2'$, and $\mathrm{d}p_3 = \mathrm{d}p_3'$. Therefore we see that

$$\mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{p} = \mathrm{d}^3\vec{x}\,'\mathrm{d}^3\vec{p}\,' \ . \tag{4.47}$$

  The phase-space element is Lorentz invariant. It follows that the phase-space density,

$$f = \frac{\mathrm{d}N}{\mathrm{d}^3\vec{x}\mathrm{d}^3\vec{p}} \ , \tag{4.48}$$

is also Lorentz invariant, since $\mathrm{d}N$ is simply the number of particles in the phase-space element.

- The specific intensity and source function. We may relate the phase-space density of photons to the specific intensity by considering the energy flux in a certain direction as

$$I_\nu \mathrm{d}\nu \mathrm{d}\Omega = c \times (h\nu f p^2 \mathrm{d}p \mathrm{d}\Omega) \ . \tag{4.49}$$

Noting that $p = h\nu/c$, we find that I_ν/ν^3 is an invariant, i.e.,

$$\frac{I'_\nu}{\nu'^3} = \frac{I_\nu}{\nu^3} \ . \tag{4.50}$$

Since the source function appears in the radiation transfer equation as the difference $(I_\nu - S_\nu)$, we may conclude that S_ν behaves like I_ν in transformation. So we also have

$$\frac{S'_\nu}{\nu'^3} = \frac{S_\nu}{\nu^3} \ . \tag{4.51}$$

- The absorption and emission coefficients. The optical depth τ is an invariant, since $e^{-\tau}$ gives the fraction of photons passing through a medium, which is a countable number. Considering radiation passing through a slab of medium of thickness ℓ at an angle θ to the slab's normal direction, the optical depth is then

$$\tau = \alpha_\nu \frac{\ell}{\cos\theta} = \nu\alpha_\nu \frac{\ell}{\nu\cos\theta} \ . \tag{4.52}$$

Let's have K' as moving parallel to the slab. Then, $\nu\cos\theta$ is the four-momentum component of the photon perpendicular to the relative motion of the frames K and K' and therefore does not change. The thickness ℓ is also the length in the perpendicular direction, so we have $\nu\alpha_v$ to be Lorentz invariant:

$$\nu'\alpha'_\nu = \nu\alpha_\nu \ . \tag{4.53}$$

The emission coefficient is $j_\nu = \alpha_\nu S_\nu$, so we have

$$\frac{j'_\nu}{\nu'^2} = \frac{j_\nu}{\nu^2} \ . \tag{4.54}$$

Chapter 5

Bremsstrahlung

In this and the next chapters we will describe emissions from high-energy charged particles. When they emit photons in the Coulomb field of an ion, the emission is called bremsstrahlung. When they do so in a magnetic field, the emission is called synchrotron radiation. High-energy charges can also produce high-energy photons by scattering photons in a photon field. This kind of scattering, or emission, is called inverse Compton scattering. The properties of these emissions can be obtained by QED calculations. Semi-classical approaches of derivation can be found in many textbooks, such as Rybicki & Lightman (1979). We will mainly describe their properties and this 'lecture notes' may serve as something like a reference book or manual for readers to easily find the needed formulae.

5.1 Bremsstrahlung of a single electron

We first consider one single electron passing through the Coulomb field of an ion of charge Ze with speed v at infinity and an impact parameter b. The bremsstrahlung spectrum for such an encounter is

$$\frac{\mathrm{d}E}{\mathrm{d}\omega}(\omega, v, b) = \begin{array}{ll} \frac{8}{3}\frac{Z^2e^6}{\pi c^3 m_\mathrm{e}^2 b^2 v^2} \ , & \text{for } \omega \ll \dfrac{v}{b} \\ 0 \ , & \text{for } \omega \gg \dfrac{v}{b} \end{array} \tag{5.1}$$

(See also Jackson (1975), p.722; Longair (2011), p.165), where ω is the angular frequency of the emission.

For a single-speed population of electrons of number density n_e and speed v, the emissivity at frequency ω in a population of ions of charge Ze and number density n_i is

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}\omega\,\mathrm{d}V\,\mathrm{d}t} &= \int_{b_\mathrm{min}}^{b_\mathrm{max}} n_\mathrm{e} v\, 2\pi b\,\mathrm{d}b\,\frac{\mathrm{d}E}{\mathrm{d}\omega}\,n_\mathrm{i} \\ &= \frac{16}{3}\frac{Z^2 e^6 n_\mathrm{e} n_\mathrm{i}}{c^3 m_\mathrm{e}^2 v}\ln\left(\frac{b_\mathrm{max}}{b_\mathrm{min}}\right) \qquad (5.2)\end{aligned}$$

The dependence on ω is weak (through b_max in the logarithm). This emissivity applies only approximately up to ω_max, where $\hbar\omega_\mathrm{max} = \frac{1}{2}m_\mathrm{e}v^2$. The determination of b_min is not trivial. See the following for the result of a more accurate QM calculation.

The emissivity for given n_i, n_e, v, and ω is

$$\frac{\mathrm{d}E}{\mathrm{d}\omega\,\mathrm{d}V\,\mathrm{d}t} = \frac{16}{3}\frac{\pi}{\sqrt{3}}\frac{Z^2 e^6 n_\mathrm{e} n_\mathrm{i}}{c^3 m_\mathrm{e}^2 v} g_\mathrm{ff}(v,\omega)\ , \qquad (5.3)$$

where

$$g_\mathrm{ff}(v,\omega) = \frac{\sqrt{3}}{\pi}\ln\Lambda \qquad (5.4)$$

is the Gaunt factor of bremsstrahlung, and

$$\begin{aligned}\ln\Lambda &= \ln\left(\frac{v_\mathrm{i}+v_\mathrm{f}}{v_\mathrm{i}-v_\mathrm{f}}\right) \\ &= \ln\left(\frac{1+\sqrt{1-\frac{\hbar\omega}{\frac{1}{2}mv_\mathrm{i}^2}}}{1-\sqrt{1-\frac{\hbar\omega}{\frac{1}{2}mv_\mathrm{i}^2}}}\right)\ , \qquad (5.5)\end{aligned}$$

where we have used $\frac{1}{2}mv_\mathrm{i}^2 = \frac{1}{2}mv_\mathrm{f}^2 + \hbar\omega$. (Longair (2011), p.167; Chiu (1968), p.223). We note that although the Gaunt factor goes to infinity at $\omega = 0$ (Figure 5.1), its integration over ω gives a finite result.

The energy loss rate of electrons is then

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{1}{n_\mathrm{e}}\int_{\hbar\omega=0}^{\hbar\omega=\frac{1}{2}mv^2}\frac{\mathrm{d}E}{\mathrm{d}\omega\,\mathrm{d}V\,\mathrm{d}t}\mathrm{d}\omega \\ &\propto n_\mathrm{i}Z^2 v \\ &\propto E^{\frac{1}{2}}\ , \qquad (5.6)\end{aligned}$$

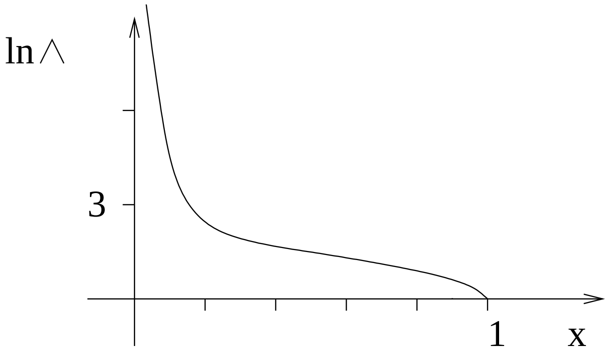

Figure 5.1: $\ln \Lambda$ in the bremsstrahlung gaunt factor. The X-axis is $x = \hbar\omega/\frac{1}{2}mv_{\mathrm{i}}^2$.

where we have taken the Gaunt factor to be roughly constant. For ultra-relativistic cases, a similar result for $\mathrm{d}E/\mathrm{d}\omega\mathrm{d}V\mathrm{d}t$ can be found (Rybicki & Lightman (1979), Eq.(5.24); Longair (2011), Eq.(6.71)). Since $v \sim c$ and $\hbar\omega_{\mathrm{max}} \sim E$, we have approximately

$$\frac{\mathrm{d}E}{\mathrm{d}t} \propto E \tag{5.7}$$

(Longair (2011), p.175). All the above are for mono-energetic electrons. In general and in practice electrons usually have an energy spectrum.

5.2 Thermal bremsstrahlung

In thermal equilibrium, electrons follow the Maxwellian distribution:

$$\mathrm{d}P \propto v^2 \exp(-\frac{mv^2}{2kT})\mathrm{d}v \ . \tag{5.8}$$

The emissivity that we discussed in the last section is then

$$\frac{\mathrm{d}E(T,\omega)}{\mathrm{d}\omega\,\mathrm{d}V\,\mathrm{d}t} = \frac{\int_{v_{\mathrm{min}}}^{\infty} \frac{\mathrm{d}E(v,\omega)}{\mathrm{d}\omega\,\mathrm{d}V\,\mathrm{d}t}\, v^2 \exp(-\frac{mv^2}{2kT})\mathrm{d}v}{\int_0^{\infty} v^2 \exp(-\frac{mv^2}{2kT})\mathrm{d}v} \ . \tag{5.9}$$

Note that since $\frac{1}{2}mv^2 \geq \hbar\omega$, we have $v_{\mathrm{min}} = \sqrt{2\hbar\omega/m}$.

After integration, we have the thermal bremsstrahlung emissivity $\varepsilon_\nu^{\mathrm{ff}} = \frac{\mathrm{d}E(T,\nu)}{\mathrm{d}\nu\,\mathrm{d}V\,\mathrm{d}t}$ as

$$
\begin{aligned}
\varepsilon_\nu^{\mathrm{ff}} &= \frac{32\pi e^6}{3m_{\mathrm{e}}c^3}\sqrt{\frac{2\pi}{3km_{\mathrm{e}}}}Z^2 n_{\mathrm{e}} n_{\mathrm{i}} T^{-\frac{1}{2}} \exp(-\frac{h\nu}{kT})\bar{g}_{\mathrm{ff}}(T,\nu) \\
&= 6.8\times 10^{-38} Z^2 n_{\mathrm{e}} n_{\mathrm{i}} T^{-\frac{1}{2}} \exp(-\frac{h\nu}{kT})\bar{g}_{\mathrm{ff}}(T,\nu) \qquad (5.10)
\end{aligned}
$$

in gaussian units. The velocity-averaged Gaunt factor, $\bar{g}_{\mathrm{ff}}(T,\nu)$, is of order of unity ($5 > \bar{g}_{\mathrm{ff}}(T,\nu) > 1$ for $10^{-4} < \frac{h\nu}{kT} < 1$). In the emissivity, the factor $T^{-\frac{1}{2}}$ comes from the $1/v$ dependence in the single electron emissivity, and the factor $\exp(-\frac{h\nu}{kT})$ comes from $v_{\min}$. The (optically thin) thermal

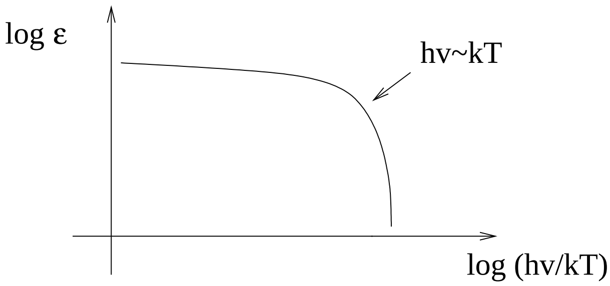

Figure 5.2: An optically-thin thermal bremsstrahlung spectrum.

bremsstrahlung spectrum is quite flat until $h\nu \sim kT$.

The total power per unit volume is

$$
\begin{aligned}
\varepsilon^{\mathrm{ff}} &= \int \varepsilon_\nu^{\mathrm{ff}} \mathrm{d}\nu \\
&= \left(\frac{2\pi k}{3m}\right)^{\frac{1}{2}} \frac{32\pi e^6}{3hmc^3} Z^2 n_{\mathrm{e}} n_{\mathrm{i}} T^{\frac{1}{2}} \bar{g}_{\mathrm{B}}(T) \\
&= 1.4\times 10^{-27} Z^2 n_{\mathrm{e}} n_{\mathrm{i}} T^{\frac{1}{2}} \bar{g}_{\mathrm{B}}(T) \qquad (5.11)
\end{aligned}
$$

in gaussian units. $\bar{g}_{\mathrm{B}}(T)$ is the frequency average of $\bar{g}_{\mathrm{ff}}(T,\nu)$. Its numerical value is between 1.1 and 1.5, so taking 1.2 should be good enough. For higher temperatures, relativistic corrections can be found as

$$
\varepsilon_{\mathrm{rel}}^{\mathrm{ff}} = \varepsilon^{\mathrm{ff}}(1 + 4.4\times 10^{-10} T/\mathrm{K}) \qquad (5.12)
$$

(Rybicki (1979), p.165).

For thermal bremsstrahlung absorption, let's consider in a thermal plasma in which only bremsstrahlung occurs. From Kirchhoff's law, we have

$$\frac{j_\nu^{\mathrm{ff}}}{\alpha_\nu^{\mathrm{ff}}} = B_\nu(T) = \frac{2\nu^2}{c^2}\frac{h\nu}{\exp(\frac{h\nu}{kT}) - 1} \; . \tag{5.13}$$

Noting that $j_\nu^{\mathrm{ff}} = \frac{\varepsilon_\nu^{\mathrm{ff}}}{4\pi}$, we get

$$\begin{aligned} \alpha_\nu^{\mathrm{ff}} &= \frac{j_\nu^{\mathrm{ff}}}{B_\nu} \\ &= \frac{4}{3}\frac{e^6}{mch}\sqrt{\frac{2\pi}{3km}} Z^2 n_{\mathrm{e}} n_{\mathrm{i}} T^{-\frac{1}{2}} \nu^{-3} (1 - \exp(-\frac{h\nu}{kT}))\bar{g}_{\mathrm{ff}}(T,\nu) \\ &= 3.7 \times 10^8 Z^2 n_{\mathrm{e}} n_{\mathrm{i}} T^{-\frac{1}{2}} \nu^{-3} (1 - \exp(-\frac{h\nu}{kT}))\bar{g}_{\mathrm{ff}}(T,\nu) \end{aligned} \tag{5.14}$$

in gaussian units. We can see that absorption is strong for low frequencies. The corresponding opacity is therefore large for those frequencies and the plasma becomes optically thick. The thermal bremsstrahlung spectrum will behave as ν^2 like the Planck function at low frequencies and then turn into an optically thin thermal bremsstrahlung spectrum at higher frequencies.

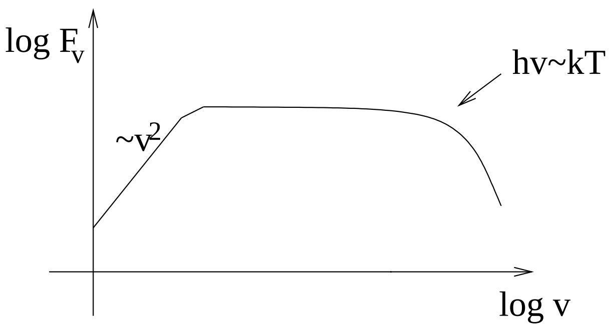

Figure 5.3: A broad-band thermal bremsstrahlung spectrum. Turning points in the spectrum may provide information about density and temperature of the emitting plasma.

With the factor of ν^{-3}, the Rosseland mean absorption coefficient is approximately like

$$\alpha_{\mathrm{R}}^{\mathrm{ff}} \propto T^{-\frac{7}{2}} Z^2 n_{\mathrm{e}} n_{\mathrm{i}} \bar{g}_{\mathrm{ff,R}}(T) \tag{5.15}$$

Noting that $\alpha = n\sigma = \rho\kappa$, we have

$$\kappa_{\mathrm{R}}^{\mathrm{ff}} \propto \rho T^{-\frac{7}{2}} \ , \tag{5.16}$$

which is called the Kramer opacity. The Rosseland mean opacity is extensively used in stellar astrophysics.

Chapter 6

Synchrotron and Curvature Radiation

6.1 Synchrotron and curvature radiation of a single electron

Emission of charged particles in a magnetic field has three names. The **cyclotron** radiation is referred to that of non-relativistic charges circulating (or spiraling) in a magnetic field and the **synchrotron** radiation is that for relativistic charges. The **curvature** radiation is the radiation of charges moving *along* curved magnetic field lines in their lowest Landau level.

The motion of a charged particle in a magnetic field.

The equation of motion reads

$$\frac{\mathrm{d}\gamma m\vec{v}}{\mathrm{d}t} = \frac{q}{c}\vec{v} \times \vec{B} \ . \tag{6.1}$$

Since the Lorentz force is always perpedicular to the velocity, if radiation loss is ignored, the energy of the particle remains constant, i.e., $\frac{\mathrm{d}\gamma}{\mathrm{d}t} = 0$. Therefore,

$$\gamma m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{q}{c}\vec{v} \times \vec{B} \ , \tag{6.2}$$

which in turn gives

$$\frac{\mathrm{d}\vec{v}_\parallel}{\mathrm{d}t} = 0 \ , \tag{6.3}$$

and

$$\gamma m \frac{\mathrm{d}\vec{v}_\perp}{\mathrm{d}t} = \frac{q}{c}\vec{v}_\perp \times \vec{B} \ . \tag{6.4}$$

The gyrofrequency and the cyclotron frequency.

The motion of a charged particle in a magnetic field can therefore be decomposed into a motion parallel to the field direction and a gyromotion around the field. The angular frequecny of this gyromotion is

$$\omega_{\mathrm{g}} = \frac{qB}{\gamma mc} \ , \tag{6.5}$$

which we call the gyration frequency, or gyrofrequency. The gyroradius r_{g} is then

$$r_{\mathrm{g}} = \frac{v_\perp}{\omega_{\mathrm{g}}} = \frac{\gamma mvc \sin\alpha}{qB} = \frac{pc}{q}\frac{\sin\alpha}{B} \ , \tag{6.6}$$

where α is the pitch angle, i.e., the angle between $\vec{v}$ and $\vec{B}$, and $\frac{pc}{q}$ is sometimes called the rigidity. We distinguish the terminology of the gyrofrequency and the cyclotron frequency by defining the latter as

$$\omega_0 = \frac{qB}{mc} \ . \tag{6.7}$$

In Rybicki & Lightman (1979) the gyrofrequency is denoted as ω_B, while in some literatures ω_B, or sometimes ω_c, is used to refer to the cyclotron frequency. To avoid this notation confusion, we use ω_{g} and ω_0 in this lecture notes.

The curvature radius and the characteristic frequency.

Consider an arc path of a charged particle's motion as shown in Figure 6.1. What we want to find is the reciprocal of the time interval of an observed light pulse, expressed in terms of the particle energy and the curvature radius or the magnetic field strength. From Figure 6.1, we have $\Delta\theta \sim \frac{2}{\gamma}$ from the

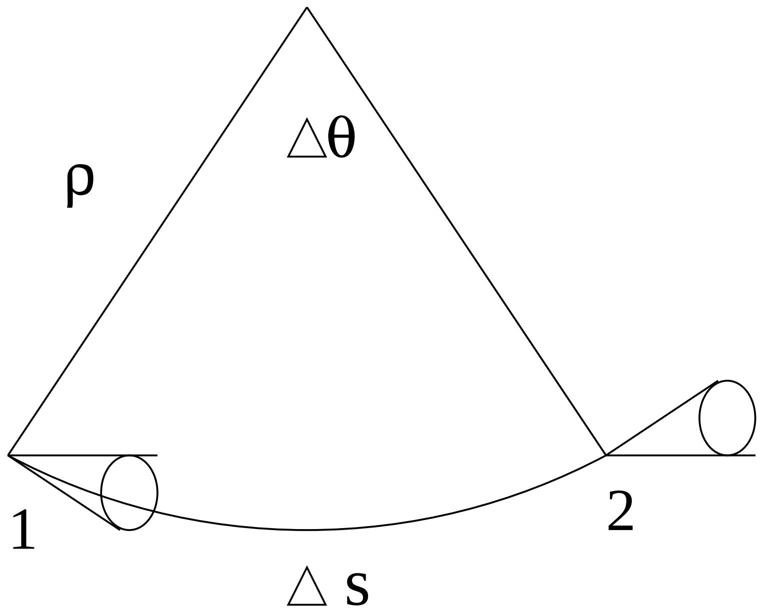

Figure 6.1: A particle moving along an arc of curvature radius ρ.

relativistic beaming effect. We have also $\rho\Delta\theta = \Delta s$ and $v\Delta t = \Delta s$, and therefore $\Delta t = \frac{2\rho}{\gamma v}$, where Δt is the time for the particle to move from point 1 to point 2.

To link the curvature radius ρ of the gyromotion with the field strength, note that $\gamma m \frac{\Delta \vec{v}}{\Delta t} = \frac{q}{c}\vec{v} \times \vec{B}$ and $|\Delta \vec{v}| = v\Delta\theta$, and then we have

$$\rho = \frac{\Delta s}{\Delta \theta} = \frac{v\Delta t}{|\Delta \vec{v}|/v} = \frac{v^2 \gamma mc}{qBv \sin\alpha} = \frac{v}{\omega_{\rm g} \sin\alpha} \ . \tag{6.8}$$

At this point, the relation between the gyroradius and the curvature radius can be found, by noting that $\omega_{\rm g} r_{\rm g} = v \sin\alpha$, to be $r_{\rm g} = \rho \sin^2\alpha$.

The time interval of the observed pulse is then

$$\begin{aligned} \Delta t_{\rm ob} &= \Delta t (1 - \frac{v}{c}) \\ &= \frac{2}{\gamma \omega_{\rm g} \sin\alpha} \frac{1}{2\gamma^2} \\ &= \frac{1}{\gamma^3 \omega_{\rm g} \sin\alpha} \ , \end{aligned} \tag{6.9}$$

where we have expressed Δt in terms of $\omega_{\rm g}$ and taken $\beta = 1 - \frac{1}{2\gamma^2}$ for relativistic particles. The γ^3 dependence comes from the beaming effect and the kinetic Doppler effect.

Conventionally a factor of three halfs is added to the definition of the characteristic frequency for synchrotron radiation:

$$\begin{aligned} \omega_{\rm c} &= \frac{3}{2}\gamma^3 \omega_{\rm g} \sin\alpha \\ &= \frac{3}{2}\gamma^2 \frac{qB}{mc} \sin\alpha \ . \end{aligned} \tag{6.10}$$

For curvature radiation, the characteristic frequency is simply

$$\omega_{\rm c} = \frac{3}{2}\gamma^3 \frac{c}{\rho} \ , \tag{6.11}$$

where the speed v is approximated with the speed of light c, as in most cases of interest.

The single-electron synchrotron radiation spectrum.

The synchrotron radiation power per frequency of a single electron in the two perpendicular linear polarization states as shown in Figure 6.2 are (Rybicki & Lightman 1979, p.179)

$$\frac{{\rm d}P_\perp}{{\rm d}\omega} = \frac{\sqrt{3}}{4\pi}\frac{q^3}{mc^2} B \sin\alpha (F(x) + G(x)) \ , \tag{6.12}$$

and

$$\frac{{\rm d}P_\parallel}{{\rm d}\omega} = \frac{\sqrt{3}}{4\pi}\frac{q^3}{mc^2} B \sin\alpha (F(x) - G(x)) \ , \tag{6.13}$$

with

$$F(x) = x \int_x^\infty K_{\frac{5}{3}}(\xi){\rm d}\xi \ , \tag{6.14}$$

$$G(x) = x K_{\frac{2}{3}}(x) \ , \tag{6.15}$$

and

$$x = \frac{\omega}{\omega_{\rm c}} \ , \tag{6.16}$$

where K's are the modified Bessel functions of order $\frac{5}{3}$ and $\frac{2}{3}$ and $\|$ and $\perp$ are for different polarizations defined in Figure 6.2. They are paralle and perpendicular to the projection of the magnetic field on the plane of the sky. This notation is opposite to that used in Jackson (1999), which takes the plane of motion, and therefore the direction of acceleration, as the reference to define $\|$ and $\perp$.

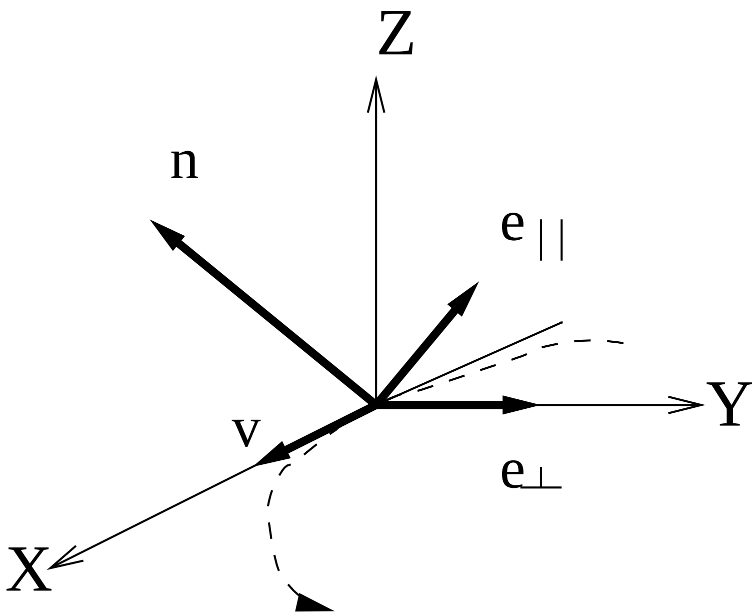

Figure 6.2: The instantaneous orbital plane is chosen to be the $X-Y$ plane with the particle velocity in the X direction. The direction towards the observer, $\hat{n}$, is chosen in the $X-Z$ plane. The directions of polarization are $\hat{e}_\perp \parallel \hat{y}$ and $\hat{e}_\parallel = \hat{n} \times \hat{e}_\perp$.

The total power spectrum is

$$\frac{\mathrm{d}P}{\mathrm{d}\omega} = \frac{\sqrt{3}}{2\pi}\frac{q^3}{mc^2} B \sin\alpha F(x) \ , \tag{6.17}$$

and

$$\begin{aligned} F(x) \sim \quad & \frac{4\pi}{\sqrt{3}\Gamma(\frac{1}{3})}\left(\frac{x}{2}\right)^{\frac{1}{3}} \ , \quad x \ll 1 \\ & \left(\frac{\pi}{2}\right)^{\frac{1}{2}} x^{\frac{1}{2}} \exp(-x) \ , \quad x \gg 1 \ . \end{aligned} \tag{6.18}$$

The function $F(x)$ is plotted in Figure 6.3.

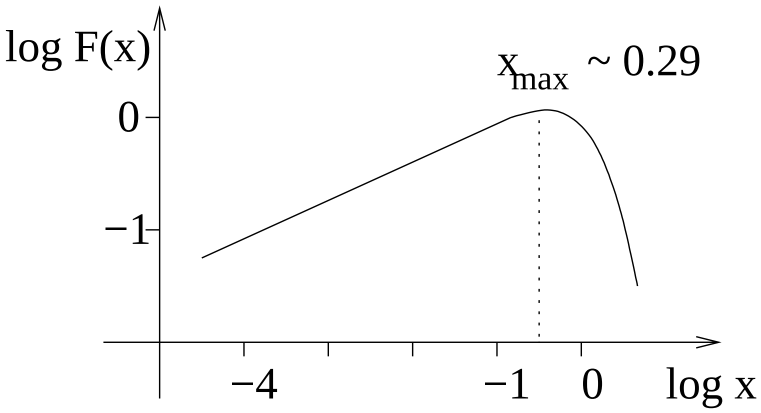

Figure 6.3: The synchrotron radiation spectrum of a single electron. $x = \omega/\omega_c$.

The total emitted power.

Integrating over the frequency, we obtain the total emitted power as

$$\begin{aligned} P &= \int \frac{dP}{d\omega} d\omega \\ &= \frac{\sqrt{3}}{2\pi} \frac{q^3}{mc^2} B \sin\alpha \int_0^\infty F(x) dx (\frac{3}{2}\gamma^3 \omega_g \sin\alpha) \\ &= \frac{2}{3}\frac{q^2}{c}\gamma^2 \omega_0^2 \sin^2\alpha \\ &= -\dot{\gamma}_{syn} mc^2 \end{aligned} \tag{6.19}$$

where $\int_0^\infty F(x)dx = \frac{8\pi}{9\sqrt{3}}$ has been employed. We note that this is the energy loss rate via synchrotron radiation, and it depends quadratically on the particle energy γ and on the field strength B. It may be expressed in terms of the changing rate of the electron's Lorentz factor, as shown in the last equality in the above.

This result can also be obtained from the relativistic version of Larmor's formula $P = \frac{2q^2}{3c^3}\gamma^4(a_\perp^2 + \gamma^2 a_\parallel^2)$ and noting that $a_\parallel = 0$ and $a_\perp = \omega_g v_\perp$, that is,

$$P = \frac{2q^2}{3c^3}\gamma^4 \omega_g^2 v_\perp^2$$

$$= \frac{2q^2}{3c}\gamma^2\omega_0^2\beta_\perp^2 \ . \tag{6.20}$$

We may further re-write this result in terms of Thomson cross section, $\sigma_{\rm T} = \frac{8\pi}{3}(\frac{e^2}{mc^2})^2$, and the magnetic field energy density, $u_{\rm B} = \frac{B^2}{8\pi}$, to get $P = 2\sigma_{\rm T}cu_{\rm B}\gamma^2\beta_\perp^2$. Noting that $\beta_\perp = \beta\sin\alpha$ and $\frac{1}{4\pi}\int\sin^2\alpha{\rm d}\Omega = \frac{2}{3}$, the averaged power, for a uniform distribution in the pitch angle, is

$$\langle P\rangle_{\rm syn} = \frac{4}{3}\sigma_{\rm T}cu_{\rm B}\gamma^2\beta^2 \ . \tag{6.21}$$

This should be compared with the inverse Compton energy loss rate $\dot\gamma_{\rm ic}mc^2 = -\frac{4}{3}\sigma_{\rm T}cu_{\rm ph}\gamma^2\beta^2$ for an isotropic incident photon field discussed in the next chapter.

The curvature radiation.

With the replacement $\frac{c}{\omega_{\rm g}\sin\alpha} \Rightarrow \rho$ we have the spectrum for curvature radiation:

$$\frac{{\rm d}P_\perp}{{\rm d}\omega} = \frac{\sqrt{3}}{4\pi}\frac{q^2}{\rho}\gamma(F(x)+G(x)) \tag{6.22}$$

$$\frac{{\rm d}P_\parallel}{{\rm d}\omega} = \frac{\sqrt{3}}{4\pi}\frac{q^2}{\rho}\gamma(F(x)-G(x)) \tag{6.23}$$

$$\frac{{\rm d}P}{{\rm d}\omega} = \frac{\sqrt{3}}{2\pi}\frac{q^2}{\rho}\gamma F(x) \tag{6.24}$$

The total power is

$$\begin{aligned} P &= \frac{\sqrt{3}}{2\pi}\frac{q^2}{\rho}\gamma\frac{8\pi}{9\sqrt{3}}\frac{3}{2}\gamma^3\frac{c}{\rho} \\ &= \frac{2}{3}\frac{q^2c}{\rho^2}\gamma^4 \\ &= -\dot\gamma_{\rm cur}mc^2 \ . \end{aligned} \tag{6.25}$$

The curvature radiation energy loss rate depends on the particle energy in the 4th power. Again, this can be obtained through $P = \frac{2q^2}{3c^3}\gamma^4(\frac{c^2}{\rho})^2$.

6.2 Synchrotron and curvature radiation of a population of electrons

The spectral index.

Consider a population of electrons with the following energy distribution:

$$N_{E,\mathrm{e}}\mathrm{d}E = C\,E^{-p}\mathrm{d}E\ ,\ E_1 < E < E_2\ . \tag{6.26}$$

From Eq.(6.17) we have the power spectrum for a population of electrons as

$$\frac{\mathrm{d}P}{\mathrm{d}\omega} \propto \int_{\gamma_1}^{\gamma_2} F(\frac{\omega}{\omega_\mathrm{c}})\,\gamma^{-p}\mathrm{d}\gamma\ , \tag{6.27}$$

in which the integration over energy is expressed in terms of the Lorentz factor, which enters the function F via the characteristic frequency ω_c. In this way one may derive the dependence of the power spectrum on frequency ω and obtain the spectral index of the resultant power-law spectrum.

One may also take another simpler approach by considering

$$P_\omega\mathrm{d}\omega \propto \dot\gamma\gamma^{-p}\mathrm{d}\gamma\ , \tag{6.28}$$

in which P_ω, as usual, is $\frac{\mathrm{d}P}{\mathrm{d}\omega}$, and a certain particle energy range $\mathrm{d}\gamma$ and its corrersponding major radiation frequency $\mathrm{d}\omega$ are considered. For synchrotron radiation, we have $\dot\gamma \propto \gamma^2$, $\omega \sim \omega_\mathrm{c} \propto \gamma^2$, and $\mathrm{d}\gamma \propto \omega^{-\frac{1}{2}}\mathrm{d}\omega$, therefore

$$P_\omega\mathrm{d}\omega \propto \omega^{-\frac{p-1}{2}}\mathrm{d}\omega\ . \tag{6.29}$$

For curvature radiation, we have $\dot\gamma \propto \gamma^4$, $\omega \sim \omega_\mathrm{c} \propto \gamma^3$, and $\mathrm{d}\gamma \propto \omega^{-\frac{2}{3}}\mathrm{d}\omega$. Therefore we have

$$P_\omega\mathrm{d}\omega \propto \omega^{-\frac{p-2}{3}}\mathrm{d}\omega\ . \tag{6.30}$$

The synchrotron and curvature radiation spectra of a power-law distribution of electrons with power index p are also a power law with power index $(p-1)/2$ and $(p-2)/3$ respectively. Note that the spectrum referred here is the one proportional to the flux density (F_ν), i.e., to energy flux per unit frequency. It is different from that of energy flux per decade of frequency (νF_ν) or photon number flux per unit energy ($\dot N_{E,\gamma} \propto F_\nu/\nu$), which are used in different occasions.

The polarization.

Synchrotron radiation from a single electron in a narrow bandwidth, as observed from a certain direction, is in general elliptically polarized with its major axis perpendicular to the magnetic field direction in the sky. For a population of electrons with a smooth distribution in pitch angles, elliptical polarization will be cancelled to become partial linear polarization in the direction perpendicular to the magnetic field in the sky. The polarization degree of this linear polarization at a certain frequency (within a narrow frequency band) for a *single-energy* population is

$$\Pi(\omega) = \frac{P_{\omega,\perp} - P_{\omega,\parallel}}{P_{\omega,\perp} + P_{\omega,\parallel}} = \frac{G(x)}{F(x)} \ , \tag{6.31}$$

whose numerical value is between 0.5 and 1.

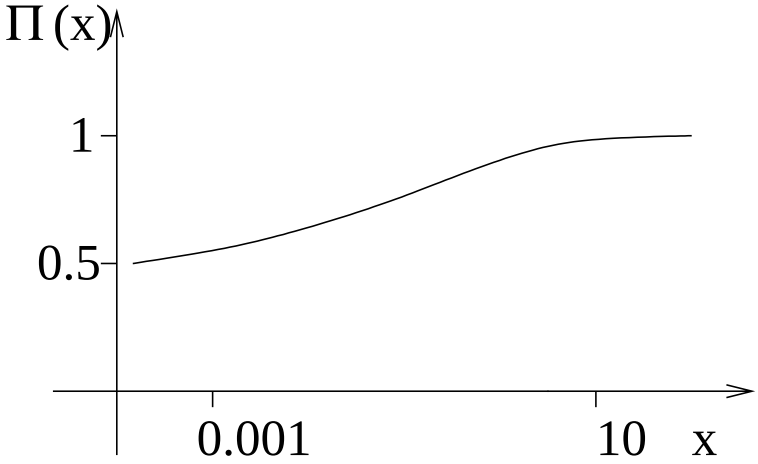

Figure 6.4: Polarization degree as a function of frequency for a single-energy population synchrotron/curvature radiation. $x = \omega/\omega_{\rm c}$.

For the radiation integrated over frequencies, noting that

$$\frac{\int_0^\infty G(x)\,{\rm d}x}{\int_0^\infty F(x)\,{\rm d}x} = \frac{3}{4} \ , \tag{6.32}$$

we therefore have

$$\Pi = \frac{P_\perp - P_\parallel}{P_\perp + P_\parallel} = 0.75 \ . \tag{6.33}$$

These results apply to both synchrotron and curvature radiation.

For a population of electrons distributed as $N_E \mathrm{d}E = CE^{-p}\mathrm{d}E$, we have for the synchrotron radiation

$$\Pi(\omega) = \frac{\int_0^\infty G(x)\gamma^{-p}\mathrm{d}\gamma}{\int_0^\infty F(x)\gamma^{-p}\mathrm{d}\gamma} \ , \tag{6.34}$$

to take into account contributions from electrons of different energy. Since $x = \omega/\omega_\mathrm{c}$ and $\omega_\mathrm{c} \propto \gamma^2$, we have, *for a fixed* ω, $\gamma \propto (\frac{\omega}{x})^{\frac{1}{2}}$ and $\mathrm{d}\gamma \propto \omega^{\frac{1}{2}} x^{-\frac{3}{2}}\mathrm{d}x$. Therefore,

$$\begin{aligned} \Pi &= \frac{\int_0^\infty G(x)\, x^{\frac{p-3}{2}}\mathrm{d}x}{\int_0^\infty F(x)\, x^{\frac{p-3}{2}}\mathrm{d}x} \\ &= \frac{p+1}{p+\frac{7}{3}} \ , \end{aligned} \tag{6.35}$$

which depends on the power index p but *not on the frequency* ω. This statement is probably not valid for a very large ω, since we have taken the integration over x up to infinity. By imposing $\gamma > \frac{\hbar\omega}{m_\mathrm{e}c^2}$, we should have $x < \frac{(m_\mathrm{e}c^2)^2}{\hbar\omega_0\hbar\omega}$, neglecting the factors of $\frac{3}{2}$ and $\sin\alpha$. For the above result to be valid, it is required that $\frac{\hbar\omega}{m_\mathrm{e}c^2} \ll \frac{m_\mathrm{e}c^2}{\hbar\omega_0}$. A so-called critical field is so defined that $\hbar\omega_0 = m_\mathrm{e}c^2$, which gives the critical field B_q as

$$B_\mathrm{q} = 4.4 \times 10^{13}\,\mathrm{G} \ . \tag{6.36}$$

Around and above this field strength, classical electrodynamics no longer provides a good description. Way below that field strength, the validity range in ω of the above result is fairly large.

For curvature radiation, similarly, since $x = \omega/\omega_\mathrm{c}$ and $\omega_\mathrm{c} \propto \gamma^3$, we have, *for a fixed* ω, $\gamma \propto (\frac{\omega}{x})^{\frac{1}{3}}$ and $\mathrm{d}\gamma \propto \omega^{\frac{1}{3}} x^{-\frac{4}{3}}\mathrm{d}x$. Therefore,

$$\begin{aligned} \Pi &= \frac{\int_0^\infty \gamma\, G(x)\gamma^{-p}\mathrm{d}\gamma}{\int_0^\infty \gamma\, F(x)\gamma^{-p}\mathrm{d}\gamma} \\ &= \frac{\int_0^\infty G(x)\, x^{\frac{p-5}{3}}\mathrm{d}x}{\int_0^\infty F(x)\, x^{\frac{p-5}{3}}\mathrm{d}x} \\ &= \frac{p+1}{p+3} \ . \end{aligned} \tag{6.37}$$

One should note the γ factor in front of G and F. Consideration of the ω validity range similar to that for the synchrotron radiation gives $\frac{\hbar\omega}{m_\mathrm{e}c^2} \ll \sqrt{\frac{m_\mathrm{e}c^2}{\hbar(\frac{3}{2}\frac{c}{\rho})}}$.

Some useful relations.

$$\int_0^\infty x^\mu F(x)\mathrm{d}x = \frac{2^{\mu+1}}{\mu+2}\Gamma(\frac{\mu}{2}+\frac{7}{3})\Gamma(\frac{\mu}{2}+\frac{2}{3}) \tag{6.38}$$

$$\int_0^\infty x^\mu G(x)\mathrm{d}x = 2^\mu\Gamma(\frac{\mu}{2}+\frac{4}{3})\Gamma(\frac{\mu}{2}+\frac{2}{3}) \tag{6.39}$$

$$\Gamma(n+1) = n\Gamma(n) \tag{6.40}$$

$$\Gamma(n) = (n-1)! \tag{6.41}$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi} \tag{6.42}$$

$$\Gamma(1) = 1 \tag{6.43}$$

The synchrotron self-absorption.

For synchrotron radiation, $F_\nu \propto \nu^{-\frac{p-1}{2}}$, the brightness temperature will be

$$\begin{aligned} kT_\mathrm{b} &= \frac{c^2}{2\nu^2}\frac{F_\nu}{\Omega} \\ &\propto \nu^{-\frac{p+3}{2}}, \end{aligned} \tag{6.44}$$

which will be very high at low frequencies. On the other hand, self-aborption is also expected to be significant by the consideration of detailed balance. For thermal radiation, the source function S_ν is proportional to the thermal temperature T of the system as $S_\nu \propto \nu^2 T$ for low frequencies. The brightness temperature T_b as derived from the specific intensity is always smaller than the system's thermal temperature. T_b approaches T only when the system is extremely optically thick. For non-thermal populations, although rigorous derivation may do better (Rybicki & Lightman (1979), page 189), we may take an anology by considering the source function to be $S_\nu \propto \nu^2 T_\mathrm{K}$, where the kinetic temperature T_k represents the particle energy, that is, $T_\mathrm{k} \propto \gamma$.

The brightness temperature approaches $T_{\rm k}$ when the system is optically thick at low frequencies. Therefore, for such low frequencies,

$$\begin{aligned} F_\nu &\propto \nu^2 T_{\rm k} \\ &\propto \nu^{+\frac{5}{2}} \ , \end{aligned} \tag{6.45}$$

where in the last step we have employed $\gamma \propto \nu^{\frac{1}{2}}$ for the major frequency (characteristic frequency) being proportional to γ^2. This behavior is sketched in Figure 6.5.

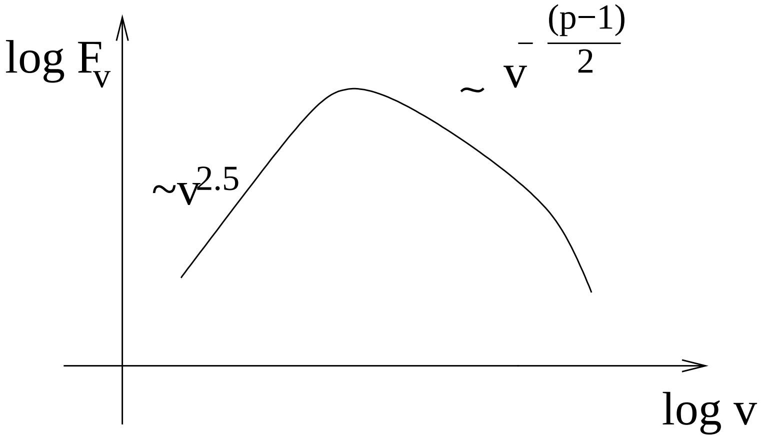

Figure 6.5: A broad-band synchrotron radiation spectrum.

Chapter 7

Compton Scattering

7.1 Compton scattering

When the incident photon energy is comparable to the electron rest energy, the discussion of Thomson scattering is no longer valid because of the quantum nature of photons. For an electron at rest, the change of the photon energy after scattering is described as the following:

$$\frac{\varepsilon_1}{\varepsilon} = \frac{1}{1+\varepsilon(1-\cos\alpha)} \ , \tag{7.1}$$

where ε is the incident photon energy normalized by the electron rest energy, i.e., $\varepsilon = \frac{h\nu}{m_{\mathrm{e}}c^2}$, ε_1 is that of the scattered photon, and α is the scattering angle. This can be re-arranged to be

$$\cos\alpha = 1 + \frac{1}{\varepsilon} - \frac{1}{\varepsilon_1} \ . \tag{7.2}$$

It can also be expressed in terms of wavelengths as

$$\frac{\Delta\lambda}{\lambda} = \frac{\lambda_1 - \lambda}{\lambda} = \varepsilon(1-\cos\alpha) \ . \tag{7.3}$$

It shows that the fractional change in wavelength is of the order of ε. The above equation can also be arranged to be

$$\Delta\lambda = \frac{h}{m_{\mathrm{e}}c}(1-\cos\alpha) \ , \tag{7.4}$$

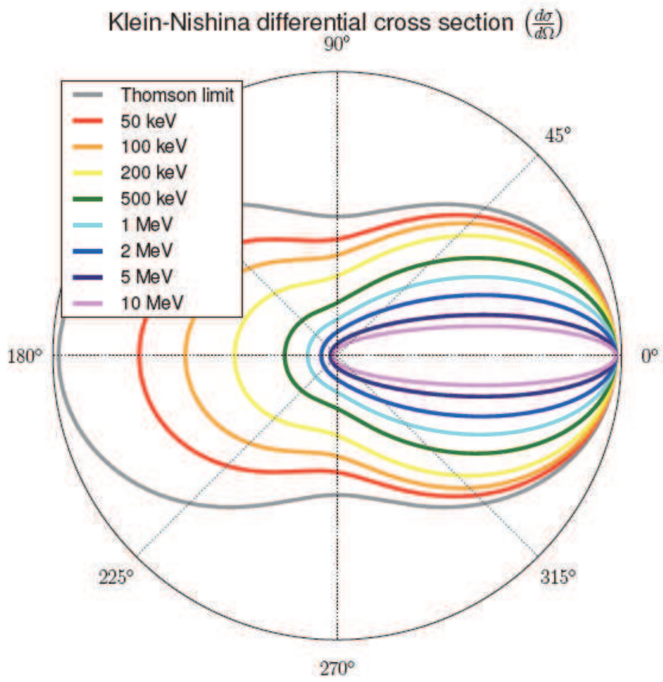

Figure 7.1: The differential Klein-Nishina cross section. (Taken from Mark Bandstra, 2010, PhD thesis, UC Berkeley)

and the factor $\frac{h}{m_e c}$ is called the Compton wavelength of the electron.

For *unpolarized* incident photons, the cross section of the scattering is the **Klein-Nishina cross section** (Heitler 1954):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{r_\mathrm{e}^2}{2}\frac{\varepsilon_1^2}{\varepsilon^2}\left(\frac{\varepsilon}{\varepsilon_1} + \frac{\varepsilon_1}{\varepsilon} - \sin^2\alpha\right) \, . \tag{7.5}$$

We note that there is also α dependence in $\varepsilon/\varepsilon_1$. This differential cross section is always smaller than that of the Thomson scattering, to which it approaches when $\varepsilon_1 \sim \varepsilon$; see Fig. 7.1.

The total cross section can be found to be

$$\sigma_{\mathrm{KN}} = \pi r_\mathrm{e}^2 \frac{1}{\varepsilon}\left((1 - \frac{2(\varepsilon+1)}{\varepsilon^2})\ln(2\varepsilon+1) + \frac{1}{2} + \frac{4}{\varepsilon} - \frac{1}{2(2\varepsilon+1)^2}\right) . \tag{7.6}$$

For $\varepsilon \ll 1$,

$$\sigma_{\mathrm{KN}} \sim \frac{8}{3}\pi r_{\mathrm{e}}^2(1-2\varepsilon) = \sigma_{\mathrm{T}}(1-2\varepsilon) \tag{7.7}$$

and for $\varepsilon \gg 1$,

$$\sigma_{\mathrm{KN}} \sim \pi r_{\mathrm{e}}^2 \frac{1}{\varepsilon}(\ln(2\varepsilon)+\frac{1}{2}) \ . \tag{7.8}$$

To get the above approximation, $\ln(1+x) \sim x - \frac{x^2}{2}$ is used for $x \ll 1$.

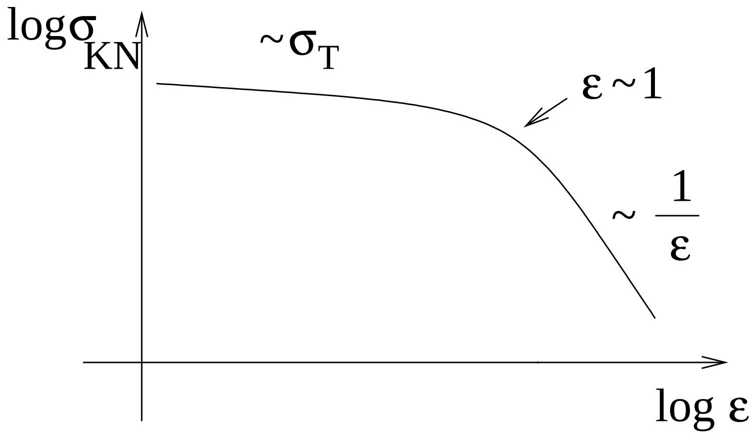

Figure 7.2: The Klein-Nishina cross section.

For *polarized* incident photons,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{r_{\mathrm{e}}^2}{2}\frac{\varepsilon_1^2}{\varepsilon^2}\left(\frac{\varepsilon}{\varepsilon_1}+\frac{\varepsilon_1}{\varepsilon}-2\sin^2\alpha\,\cos^2\eta\right) \ , \tag{7.9}$$

where η is the angle between the polarization of the incident photon and the scattering plane. One can see that the scattering is preferred in the direction of $\eta = \frac{\pi}{2}$ (Lei et al., 1997). This property may be employed to measure the polarization of incoming photons in Compton telescopes.

In consideration of detector designs, the recoil energy of electrons, which will be deposited in the detector and turned into electric signals, is more relevant. The electron recoil energy is

$$\begin{aligned} E_{\mathrm{e}} &= m_{\mathrm{e}}c^2(\varepsilon-\varepsilon_1) \\ &= m_{\mathrm{e}}c^2\varepsilon(1-\frac{1}{1+\varepsilon(1-\cos\alpha)}) \ . \end{aligned} \tag{7.10}$$

The minimum recoil energy is zero when $\alpha = 0$, and the maximum is $E_{\rm e} = m_{\rm e}c^2\varepsilon(\frac{2\varepsilon}{1+2\varepsilon})$ when $\alpha = \pi$. The gap between the incident photon energy and the maximum electron recoil energy, i.e., the energy of the back-scattered photon, is then

$$\begin{aligned}\varepsilon_{\rm b} &= \varepsilon - E_{\rm e,max}/m_{\rm e}c^2 \\ &= \varepsilon(\frac{1}{1+2\varepsilon}) \\ &\approx \frac{1}{2} \ ,\end{aligned} \tag{7.11}$$

where the last approximation is for $2\varepsilon \gg 1$ (but note that we always have $\mathrm{d}\varepsilon_{\rm b}/\mathrm{d}\varepsilon > 0$). For very high energy incident photons, $E_{\rm e,max}$ approaches the energy of the incident photon.

7.2 Inverse Compton scattering

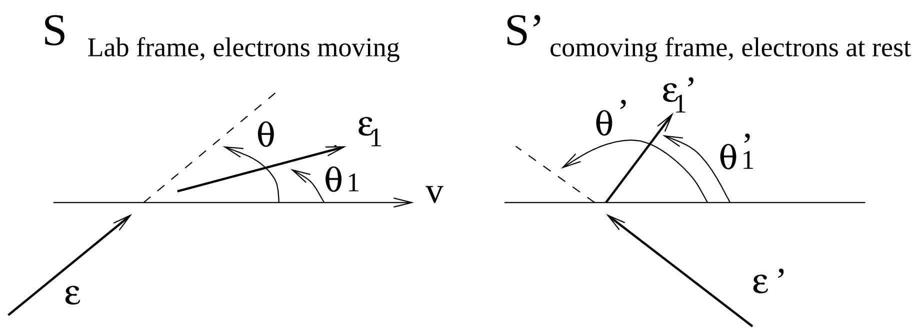

Figure 7.3: Notations used in analyzing inverse Compton scattering.

If electrons are very energetic, photons after scattering tend to gain energy, instead of losing. We may take the approach similar to what is usually done for Fermi accelaration to consider the photon energy in the co-moving frame of electrons and then transform that back to the observer's frame; see Figure 7.3 and notations defined therein. Recall that a photon's energy-momentum four vector is $(k_0, \vec{k})$, where $k_0 = \varepsilon/c$ and $|\vec{k}| = k_0$. It is clear that

$$\varepsilon' = \varepsilon\gamma(1 - \beta\cos\theta) \ , \tag{7.12}$$

$$\varepsilon_1' = \varepsilon' \frac{1}{1+\varepsilon'(1-\cos\alpha')} , \tag{7.13}$$

and

$$\varepsilon_1 = \varepsilon_1' \gamma (1+\beta\cos\theta_1') , \tag{7.14}$$

therefore

$$\varepsilon_1 = \frac{\varepsilon\gamma^2(1-\beta\cos\theta)(1+\beta\cos\theta_1')}{1+\varepsilon\gamma(1-\beta\cos\theta)(1-\cos\alpha')} . \tag{7.15}$$

We then have

$$\varepsilon_1 \sim \gamma^2\varepsilon \text{ , for } \varepsilon\gamma \ll 1 . \tag{7.16}$$

The photon energy is boosted by a factor of γ^2. The highest energy that a photon can reach in this regime is $\varepsilon_1 = 4\gamma^2\varepsilon$ for the case of head-on collision ($\theta = \pi$, $\theta_1' = 0$). On the other hand,

$$\varepsilon_1 \sim \gamma \text{ , for } \varepsilon\gamma \gg 1 . \tag{7.17}$$

In such a case electrons give almost all the energy to photons. But note that it is also possible that, when $\varepsilon > \gamma$, photons in fact give energy to electrons. The scattering cross section for $\varepsilon\gamma \gg 1$ is smaller than that for $\varepsilon\gamma \ll 1$.

The inverse-Compton emitted power

We now consider an electron colliding with a distribution of photons to lose energy via inverse Compton scattering. To derive the emitted power, one may in principle consider the emitted spectrum and integrate that over all frequencies. That may, however, depend on the specific distribution of the photon bath. Instead, in the following we take another approach to obtain a more general result. Let f be the phase space distribution function of photons, that is, $\mathrm{d}N = f\mathrm{d}^3p\mathrm{d}^3x$, which is a Lorentz invariant (because phase space volume is invarint; see Rybicki & Lightman 1979, p.145). The differential number density of photons at a certain energy ε is then

$$\mathrm{d}n = f\mathrm{d}^3p = g_\varepsilon \mathrm{d}\varepsilon . \tag{7.18}$$

Since $\mathrm{d}n/\varepsilon$ is invariant (Blumenthal & Gould 1970; see also the notes below), we have

$$\frac{g_\varepsilon \mathrm{d}\varepsilon}{\varepsilon} = \frac{g'_\varepsilon \mathrm{d}\varepsilon'}{\varepsilon'} \tag{7.19}$$

for the ambient photon field as described in different inertial frames.

The total power emitted by an electron in its rest frame is

$$\frac{\mathrm{d}E'_1}{\mathrm{d}t'} = c\sigma_\mathrm{T} \int \varepsilon' g'_\varepsilon \mathrm{d}\varepsilon' \ , \tag{7.20}$$

where we have restricted ourselves to the case of $\varepsilon\gamma \ll 1$ so that the Thomson cross section can be used and the energy change in the electron's rest frame is also neglected, i.e., that $\varepsilon'_1 = \varepsilon'$ has been adopted. Note that the emitted power for a front-back symmetric radiation pattern (in the instantaneous rest frame of the radiating particle) is Lorentz invariant (Eq.(4.24)). We then have

$$\begin{aligned}
\frac{\mathrm{d}E_1}{\mathrm{d}t} &= c\sigma_\mathrm{T} \int \varepsilon'^2 \frac{g'_\varepsilon \mathrm{d}\varepsilon'}{\varepsilon'} \\
&= c\sigma_\mathrm{T} \int \varepsilon'^2 \frac{g_\varepsilon \mathrm{d}\varepsilon}{\varepsilon} \\
&= c\sigma_\mathrm{T}\gamma^2(1-\beta\cos\theta)^2 \int \varepsilon g_\varepsilon \mathrm{d}\varepsilon \ .
\end{aligned} \tag{7.21}$$

The last integral is in fact the photon energy density u_ph. In this equation, θ is the angle between velocities of the electron and incident photons.

The inverse Compton energy loss rate in an isotropic distribution of photons

The average of $(1-\beta\cos\theta)^2$ over all directions is $1+\frac{1}{3}\beta^2$. We therefore have

$$\frac{\mathrm{d}E_1}{\mathrm{d}t} = c\sigma_\mathrm{T}\gamma^2(1+\frac{1}{3}\beta^2)u_\mathrm{ph} \tag{7.22}$$

For the energy loss rate of electrons, we should take into account the original energy carried by the incident photons per unit time, which is simply $c\sigma_\mathrm{T}u_\mathrm{ph}$. So, the energy loss rate is

$$\begin{aligned}
P_\mathrm{ic} &= c\sigma_\mathrm{T}(\gamma^2(1+\frac{1}{3}\beta^2)-1)u_\mathrm{ph} \\
&= \frac{4}{3}\sigma_\mathrm{T}c\gamma^2\beta^2u_\mathrm{ph} \ .
\end{aligned} \tag{7.23}$$

This should be compared with $P_{\rm syn}$.

The above discussion is good in the Thomson limit, that is, for the case that Thomson scattering is a good approximation in the electron co-moving frame, $\gamma\varepsilon \ll 1$. On the other hand, if $\varepsilon_1' < \varepsilon'$, i.e., $\varepsilon' \approx \gamma\varepsilon \geq 1$, the energy loss rate is found to be

$$P_{\rm ic} = \frac{4}{3}\sigma_{\rm T} c\gamma^2\beta^2 u_{\rm ph}\left(1 - \frac{63}{10}\frac{\gamma\langle\varepsilon^2\rangle}{\langle\varepsilon\rangle}\right) \ , \tag{7.24}$$

where $\langle\varepsilon^2\rangle = \int \varepsilon^2 g_\varepsilon {\rm d}\varepsilon$ (Blumenthal & Gould 1970). In such a case, electrons can either gain or lose energy.

Now let's consider a population of electrons. If the electrons have a certain distribution with the number density being ${\rm d}n_{\rm e} = N_{\rm e}(\gamma){\rm d}\gamma$, the total energy loss rate per unit volume is just

$$P = \int P_{\rm ic} N_{\rm e}(\gamma){\rm d}\gamma \ . \tag{7.25}$$

If the electrons are in a non-relativistic thermal distribution, we have $\gamma \approx 1$, $\langle\beta^2\rangle = \langle\frac{v^2}{c^2}\rangle = \frac{3kT}{mc^2}$, and the energy loss rate per unit volume is

$$P = \left(\frac{4kT}{mc^2}\right) c\sigma_{\rm T} n_{\rm e} u_{\rm ph} \ . \tag{7.26}$$

This equation says that, averagely speaking, the ratio of electron's energy loss to the incident photon energy is $\left(\frac{4kT}{mc^2}\right)$. This is only valid for photons with energy smaller than kT, because we have neglected possible energy gain of electrons when making the assumption that $\varepsilon_1' = \varepsilon'$.

The inverse Compton scattering spectrum

For an isotropic, monochromatic incident photon field at energy ε and number density $n_{\rm ph}$ scattered off an electron of $\gamma \gg 1$ in the Thomson limit ($\gamma\varepsilon \ll 1$), the inverse Compton scattering spectrum (energy per unit time per unit energy) is

$$\frac{{\rm d}P}{{\rm d}\varepsilon_1} = 3\sigma_{\rm T} c n_{\rm ph} x f(x) \ , \tag{7.27}$$

in which $x = \frac{\varepsilon_1}{4\gamma^2\varepsilon}$ and $f(x) = 2x\ln x + x + 1 - 2x^2$ (Blumenthal & Gould 1970; Note that $\frac{{\rm d}P}{{\rm d}\varepsilon_1}$ is equal to $j(\varepsilon_1)\times\frac{\varepsilon_1}{N}$ in Eq.(7.26a) of Rybicki & Lightman

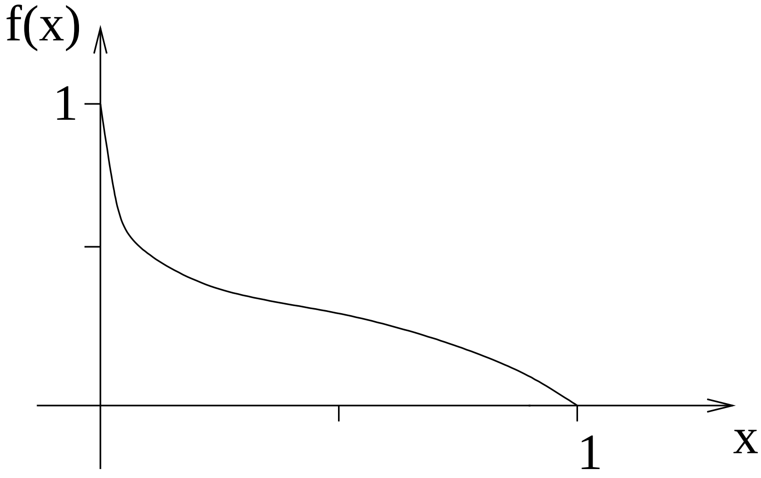

Figure 7.4: The function $f(x)$ in the single electron inverse Compton spectrum.

(1979)). The function $f(x)$ and the inverse Compton scattering spectrum of a single electron in an isotropic single-energy photon field are plotted in Figures 7.4 and 7.5, respectively.

For a distribution of photons $\mathrm{d}n_{\mathrm{ph}} = n_{\mathrm{ph},\varepsilon}\mathrm{d}\varepsilon$ scattered off electrons of a power-law distribution $\mathrm{d}n_{\mathrm{e}} \propto \gamma^{-p}\mathrm{d}\gamma$, still isotropic and in the Thomson limit, the emissivity is

$$\frac{\mathrm{d}P}{\mathrm{d}V\mathrm{d}\varepsilon_1} \propto \int n_{\mathrm{ph},\varepsilon}\gamma^{-p}xf(x)\mathrm{d}\varepsilon\mathrm{d}\gamma \ . \tag{7.28}$$

Since $x = \frac{\varepsilon_1}{4\gamma^2\varepsilon}$, we have $\gamma = \frac{1}{2}(\frac{\varepsilon_1}{\varepsilon})^{\frac{1}{2}}x^{-\frac{1}{2}}$ and $\mathrm{d}\gamma = -\frac{1}{4}(\frac{\varepsilon_1}{\varepsilon})^{\frac{1}{2}}x^{-\frac{3}{2}}\mathrm{d}x$ for fixed $\frac{\varepsilon_1}{\varepsilon}$.

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}V\mathrm{d}\varepsilon_1} &\propto \varepsilon_1^{-\frac{p-1}{2}} \int n_{\mathrm{ph},\varepsilon}\varepsilon^{\frac{p-1}{2}}\mathrm{d}\varepsilon \int_{\frac{\varepsilon_1}{4\gamma_{\max}^2\varepsilon}}^{\frac{\varepsilon_1}{4\gamma_{\min}^2\varepsilon}} x^{\frac{p-1}{2}} f(x)\mathrm{d}x \\ &\propto \varepsilon_1^{-\frac{p-1}{2}} \ . \end{aligned} \tag{7.29}$$

In the above we have assumed that the integration over x is essentially from 0 to 1 (because $\gamma_{\min} \ll \gamma_{\max}$) and therefore does not depend on $\varepsilon_1/\varepsilon$. The power index of this spectrum is the same as that for the synchratron radiation.

For a thermal photon field,

$$n_{\mathrm{ph},\varepsilon}\mathrm{d}\varepsilon = B_\nu \mathrm{d}\nu \times \frac{4\pi}{c\varepsilon} \ , \tag{7.30}$$

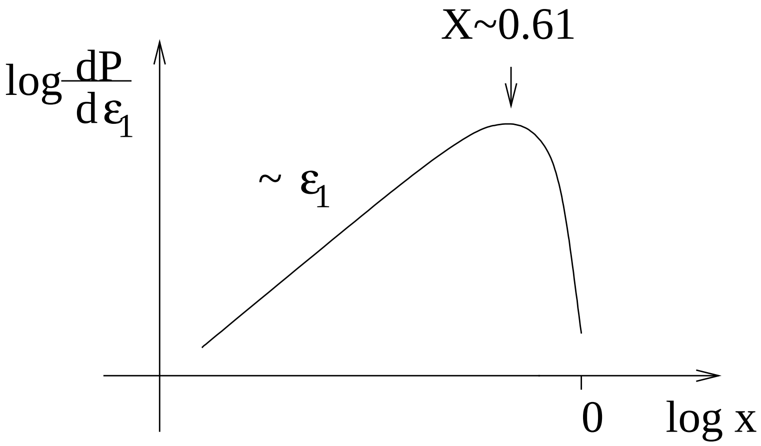

Figure 7.5: The single electron inverse Compton spectrum.

that is,

$$n_{\mathrm{ph},\varepsilon} = \frac{8\pi}{h^3c^3}\frac{\varepsilon^2}{\exp(\frac{\varepsilon}{kT})-1} \ . \tag{7.31}$$

We can see that

$$\frac{\mathrm{d}P}{\mathrm{d}V\mathrm{d}\varepsilon_1} \propto (kT)^{\frac{p+5}{2}}\varepsilon_1^{-\frac{p-1}{2}} \ , \tag{7.32}$$

that is, the emissivity depends on the photon temperature to the $\frac{p+5}{2}$ power. More detailed and more general consideration about scattering of an isotropic photon field off a population of electrons can be found in Jones (1968).

Notes to some useful Lorentz invariants

We have employed some Lorentz invariants in the above discussion. Here is a summary:

- A four-volume, i.e., a differential volume element in a four-vector space, is Lorentz invariant (Shultz 1985, p.157; Weinberg 1972, p.99).

- A phase space element, i.e., $\mathrm{d}^3x\mathrm{d}^3p$, is Lorentz invariant (Rybicki & Lightman 1979, p.145).

- The emitted power for the case of front-back symmetric radiation pattern in the rest frame of the radiating charge is invariant (Rybicki & Lightman 1979, p.139). By 'front-back symmetric' we mean for any direction of emission there is an equal probability of emission in the opposite direction. The dipole radiation adopted in Thomson scattering is such a case.

- The phase space density $\frac{\mathrm{d}N}{\mathrm{d}^3x\mathrm{d}^3p}$ is also invariant.

- The ratio of differential number density to its corresponding energy, $\mathrm{d}n/\varepsilon$, is invariant. This can be seen by noting that

$$\mathrm{d}n = \frac{\mathrm{d}N}{\mathrm{d}^3x} = \frac{\mathrm{d}N}{\mathrm{d}^4x}\mathrm{d}x_0 \ , \tag{7.33}$$

and therefore $\mathrm{d}n$ transforms like $\mathrm{d}x_0$, the time componemt of a differential displacement four-vector. Since the differential displacement four-vector is 'parallel' to its momentum four-vector ($\frac{\mathrm{d}x_i}{\mathrm{d}x_0} = \frac{p_i}{p_0}$), we have

$$\begin{aligned} \frac{\mathrm{d}x_0}{p_0} &= \frac{\sum \Lambda'_{0\mu}\mathrm{d}x'_\mu}{\sum \Lambda'_{0\mu}p'_\mu} \\ &= \frac{\sum \Lambda'_{0\mu}\mathrm{d}x'_\mu}{\sum \Lambda'_{0\mu}\frac{\mathrm{d}x'_\mu}{\mathrm{d}x'_0}p'_0} \\ &= \frac{\mathrm{d}x'_0}{p'_0} \ . \end{aligned} \tag{7.34}$$

Noting that $p_0 = \varepsilon$, we then have $\mathrm{d}n/\varepsilon$ being Lorentz invariant (Blumenthal & Gould 1970).

7.3 Comptonization

The scattering between photons and electrons brings energy exchange between these two species. Very often we are interested in how the photon distribution is altered by a certain distribution of electrons through multiple scatterings. Such an action of changing the photon distribution is called

Comptonization. In the following we will discuss the concept of the Compton optical depth and Comptonization in a thin medium and in a thermal medium.

The Compton y parameter (the Compton optical depth)

Let's first discuss how to parameterize the significance of Comptonization of a medium. Consider repeated scattering of low-energy photons in a finite medium. One may define a Compton y parameter as the following to describe how significantly a photon's energy is changed when it travels through the medium: $y :=$ (average fractional energy change per scattering $\Delta\varepsilon/\varepsilon$) $\times$ (average number of scattering in the medium).

For $\varepsilon \ll 1$, the energetics of Compton scattering off an electron at rest, Eq.(7.1), can be turned into

$$\frac{\varepsilon_1}{\varepsilon} = 1 - \varepsilon(1 - \cos\alpha) \ , \tag{7.35}$$

and after averaging over the scattering angle α we obtain the fractional energy change of the photon as

$$\frac{\Delta\varepsilon}{\varepsilon} = \frac{\varepsilon_1 - \varepsilon}{\varepsilon} = -\varepsilon \ . \tag{7.36}$$

When electrons are not strictly at rest, for example with thermal motions, photons may also gain energy, instead of losing. To the lowest order, we may describe the fractional energy change of photons in a single scattering with a linear combination of two small terms, ε and $kT/m_{\mathrm{e}}c^2$, in the following way:

$$\frac{\Delta\varepsilon}{\varepsilon} = -\varepsilon + a\frac{kT}{m_{\mathrm{e}}c^2} \ , \tag{7.37}$$

where a is a coefficient to be determined. Let's assume photons are in thermal equilibrium with electrons and the interaction between them is scattering only. In such a case photons will follow Bose-Einstein distribution with a certain chemical potential, instead of the Planck function, because the photon number is conserved. Let's further assume the number density of photons and electrons is so low that a classical description applies, i.e, photons also follow the Maxwell-Boltzmann distribution:

$$\frac{\mathrm{d}N}{\mathrm{d}\varepsilon} \propto \varepsilon^2 e^{-\varepsilon m_{\mathrm{e}}c^2/kT} \ . \tag{7.38}$$

Noting that

$$\langle\varepsilon\rangle = \frac{\int \varepsilon \frac{\mathrm{d}N}{\mathrm{d}\varepsilon}\mathrm{d}\varepsilon}{\int \frac{\mathrm{d}N}{\mathrm{d}\varepsilon}\mathrm{d}\varepsilon} = \frac{3kT}{m_{\mathrm{e}}c^2} \tag{7.39}$$

and

$$\langle\varepsilon^2\rangle = 12(\frac{kT}{m_{\mathrm{e}}c^2})^2 \ , \tag{7.40}$$

we have

$$\begin{aligned} \langle\Delta\varepsilon\rangle &= -\langle\varepsilon^2\rangle + a\frac{kT}{m_{\mathrm{e}}c^2}\langle\varepsilon\rangle \\ &= 3(\frac{kT}{m_{\mathrm{e}}c^2})^2(a-4) \ . \end{aligned} \tag{7.41}$$

With the requirement of $\langle\Delta\varepsilon\rangle = 0$ in equilibrium, we see that $a = 4$. Therefore, for non-relativistic electrons in thermal equilibrium, the average energy transfer per scattering is (Rybicki & Lightman 1979, p.209)

$$\frac{\Delta\varepsilon_{\mathrm{NR}}}{\varepsilon} = \frac{4kT}{m_{\mathrm{e}}c^2} - \varepsilon \ . \tag{7.42}$$

This equation describes the fractional energy change of a photon with energy ε in a thermal electron bath. The 'average' embedded is meant to be over the electron distribution. We in fact may also reach the above equation with Eq.(7.26) and Eq.(7.36). The $4kT$ term is the average fractional energy loss of a thermal electron via inverse Compton scattering when $\varepsilon \ll kT/m_{\mathrm{e}}c^2$ and $\gamma \sim 1$. The average fractional energy gain of an electron via Compton scattering is simply ε if $\varepsilon \ll 1$ and electrons are at rest. A linear combination of these two terms should still lead to the original results in limiting cases. Therefore they should be just summed together. We note that Eq.(7.42) <u>does not</u> require $\varepsilon \ll kT/m_{\mathrm{e}}c^2$. If now we further assume $\varepsilon \ll \frac{4kT}{m_{\mathrm{e}}c^2}$, we will have the average photon energy change per scattering in a thermal electronic gas to be

$$\frac{\Delta\varepsilon_{\mathrm{NR}}}{\varepsilon} = \frac{4kT}{m_{\mathrm{e}}c^2} \ . \tag{7.43}$$

For ultra-relativistic cases in the Thomson limit ($\gamma \gg 1$, $\gamma\varepsilon \ll 1$), we have

$$\frac{\Delta\varepsilon_{\rm UR}}{\varepsilon} = \frac{4}{3}\gamma^2 \ , \tag{7.44}$$

which can be obtained from Eq.(7.23). If the electrons are in thermal equilibrium, we have

$$\langle\gamma^2\rangle = \frac{\langle E^2\rangle}{(m_{\rm e}c^2)^2} = 12\left(\frac{kT}{m_{\rm e}c^2}\right)^2 \ , \tag{7.45}$$

and

$$\frac{\Delta\varepsilon_{\rm UR}}{\varepsilon} \sim 16\left(\frac{kT}{m_{\rm e}c^2}\right)^2 \ . \tag{7.46}$$

Therefore, for a thermal distribution of electrons, the Compton y parameter for non-relativistic and ultra-relativistic cases are

$$y_{\rm NR} = \frac{4kT}{m_{\rm e}c^2}\,{\rm Max}(\tau_{\rm es}, \tau_{\rm es}^2) \tag{7.47}$$

$$y_{\rm UR} = (\frac{4kT}{m_{\rm e}c^2})^2\,{\rm Max}(\tau_{\rm es}, \tau_{\rm es}^2) \ , \tag{7.48}$$

where $\tau_{\rm es} \sim \rho\kappa_{\rm es}R$, R is the dimension of the medium and $\kappa_{\rm es} = \frac{\sigma_{\rm T}}{m_{\rm p}} = 0.40$ cm^2 g^{-1}. The y-parameter is the Compton optical depth, which indicates the significance level of Comptonization in a medium.

Optically thin media: power-law spectra due to repeated scattering

When the mean amplification of photon energy per scattering is independent of the photon energy, i.e.,

$$\varepsilon_1 = A\varepsilon \ , \tag{7.49}$$

it is possible to result in a power-law emergent spectrum. Considering that $\varepsilon_1 = \varepsilon + \Delta\varepsilon = \varepsilon(1 + \frac{4}{3}\langle\gamma^2\rangle) = \varepsilon(1 + (\frac{4kT}{mc^2})^2) \approx \varepsilon(\frac{4kT}{mc^2})^2$ for an ultra-relativistic case in the Thomson limit, i.e., $\varepsilon \ll 1/\gamma$, and $\varepsilon_1 \sim \varepsilon(1 + \frac{4kT}{mc^2})$ for a non-relativistic case with $\varepsilon \ll kT/m_{\rm e}c^2$, we have A being independent of ε for both cases. The photon energy after k times scatterings is expected to be

$$\varepsilon_k \sim \varepsilon_{\rm i}A^k \tag{7.50}$$

If the medium's scattering optical depth is small, and the absorption optical depth is even much smaller, the probability for a photon to have k times scatterings before escaping from the medium is about $\tau_{\rm es}^k$. We therefore have

$$N(\varepsilon_k)\mathrm{d}\varepsilon_k = N(\varepsilon_{\rm i})\mathrm{d}\varepsilon_{\rm i}\tau_{\rm es}^k \ . \tag{7.51}$$

Noting that we may express $\tau_{\rm es}^k$ in terms of $\varepsilon_k/\varepsilon_{\rm i}$ as

$$\tau_{\rm es}^k = \left(\frac{\varepsilon_k}{\varepsilon_{\rm i}}\right)^{-\mu} \ , \tag{7.52}$$

where

$$\mu = \frac{-\ln \tau_{\rm es}}{\ln A} \ , \tag{7.53}$$

and $\mathrm{d}\varepsilon_k = A^k \mathrm{d}\varepsilon_{\rm i} = (\varepsilon_k/\varepsilon_{\rm i})\mathrm{d}\varepsilon_{\rm i}$, we then have

$$N(\varepsilon_k) = N(\varepsilon_{\rm i})\left(\frac{\varepsilon_k}{\varepsilon_{\rm i}}\right)^{-1-\mu} \ . \tag{7.54}$$

We see that a power-law spectrum can be produced by repeated scattering even when the electron distribution is not a power law. This result is similar to that in the Fermi mechanism, or shock acceleration. Note that the above is only valid for $\varepsilon_k \ll kT/m_{\rm e}c^2$ (NR) or $\varepsilon_k \ll m_{\rm e}c^2/kT$ (UR), and this is a photon number spectrum. For the photon energy spectrum (energy flux density spectrum), the power is simply $-\mu$.

Evolution of the photon spectrum in a non-relativistic thermal medium: the Kompaneets equation

A general discussion of repeated Compton scattering and the resultant spectra is complicated, when different Compton optical depth, different photon energy ranges, and other emissions from the media are involved. **The Kompaneets equation** is an equation to describe Comptonization in a non-relativistic thermal electron gas, which we will briefly discuss in the following.

The Kompaneets equation describes the evolution of photon occupation numbers. The occupation number n, or the phase space distribution function, is related to the phase space number density as

$$\frac{\mathrm{d}N}{\mathrm{d}^3x\mathrm{d}^3p} = \frac{g}{h^3}\, n \ , \tag{7.55}$$

where g is the statistical weight. For photons $g = 2$. The photon occupation number is related to the specific intensity by

$$n = \frac{I_\nu c^2}{2h\nu^3} \ , \tag{7.56}$$

where $h\nu$ is the photon energy. The above equation can be obtained with hints from the Planck function. In equilibrium, n is simply the Bose-Einstein distribution with zero chemical potential, and the relation between n and I_ν does not depend on whether it is in equilibrium or not. The assumption of isotropy is implicitly included, though. Otherwise, the specific intensity should be better replaced with the mean specific intensity. From Eq.(1.31), Eq.(1.32) and Eq.(1.39), we have

$$\frac{\mathrm{d}n}{\mathrm{d}s} = \frac{h\nu\phi(\nu)}{4\pi} B_{21}[-n_1 n + n_2(1+n)] \ , \tag{7.57}$$

where n_1 and n_2 are number density of systems at level 1 and 2. From this example we see that the occupation number n in the term $(1+n)$ stands for stimulated processes.

The Boltzmann equation for n is, due to scatterings with a population of electrons,

$$\begin{aligned} \frac{\partial n(\omega)}{\partial t} &= c\int \mathrm{d}^3p \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega(f_{\mathrm{e}}(\vec{p}_1)n(\omega_1)(1+n(\omega)) \\ &\quad -f_{\mathrm{e}}(\vec{p})n(\omega)(1+n(\omega_1))) \end{aligned} \tag{7.58}$$

where $f_{\mathrm{e}}(\vec{p})$ and $f_{\mathrm{e}}(\vec{p}_1)$ are electrons' phase space number density, ω_1 depends on $\vec{p}$ or $\vec{p}_1$ to accomplish the scattering, and $n(\omega) = n(\nu)$. The first term at the right hand side is for the increase of $n(\omega)$ due to scattering between electrons of momentum $\vec{p}_1$ and photons of frequency ω_1. The second term describes the decrease. Stimulated processes are included via n in $(1+n)$. This equation is in general difficult and complicated to solve. If the electrons are non-relativistic, the fractional energy change per scattering is small. The Boltzmann equation can be expanded to the second order in this small quantity. This approximation leads to the Fokker-Planck equation. The Fokker-Planck equation for photons scattering off a non-relativistic, thermal distribution of electrons is called the Kompaneets equation. We define the photon energy change with a dimensionless quantity Δ,

$$\Delta = \frac{\hbar(\omega_1 - \omega)}{kT} \ . \tag{7.59}$$

For the case of $\Delta \ll 1$, we can expand n to be

$$n(\omega_1) = n(\omega) + (\omega_1 - \omega)\frac{\partial n}{\partial \omega} + \frac{1}{2}(\omega_1 - \omega)^2\frac{\partial^2 n}{\partial^2 \omega} + ... \tag{7.60}$$

Then, the Kompaneets equation is

$$\begin{aligned}
&\frac{1}{c}\frac{\partial n}{\partial t} \\
&= (n' + n(1+n))\int\int \mathrm{d}^3p\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega f_{\mathrm{e}}\Delta \\
&+(\frac{1}{2}n'' + n'(1+n) + \frac{1}{2}n(1+n))\int\int \mathrm{d}^3p\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega f_{\mathrm{e}}\Delta^2 \ ,
\end{aligned} \tag{7.61}$$

where $n' = \partial n/\partial x$ and $x = \frac{\hbar\omega}{kT}$. This can be turned into the following form (Rybicki & Lightman 1979, p.214-215):

$$\frac{\partial n}{\partial t_{\mathrm{c}}} = \left(\frac{kT}{m_{\mathrm{e}}c^2}\right)\frac{1}{x^2}\frac{\partial}{\partial x}(x^4(n' + n + n^2)) \ , \tag{7.62}$$

with $t_{\mathrm{c}} = (n_{\mathrm{e}}\sigma_{\mathrm{T}}c)t$ is the time in units of mean time between scatterings.

Eq.(7.62) can be solved numerically in general. If the system is very optically thick, the photon spectrum reaches equilibrium after a sufficient number of scatterings. We note that the Bose-Einstein distribution ($n = (\exp(x+\alpha) - 1)^{-1}$) makes $n' + n + n^2 = 0$. On the other hand, if before reaching equilibrium, photons already escape from the system, one should include input and the escape of photons in the Kompaneets equation, that is,

$$\frac{\partial n}{\partial t_{\mathrm{c}}} = \left(\frac{kT}{mc^2}\right)\frac{1}{x^2}\frac{\partial}{\partial x}(x^4(n' + n + n^2)) + Q(x) - \frac{n}{\mathrm{Max}(\tau_{\mathrm{es}}, \tau_{\mathrm{es}}^2)} \ , \tag{7.63}$$

where $Q(x)$ is the source term and the escape probability per scattering is taken to be the reciprocal of the average number of scattering in the medium. Consider a steady-state solution and that Q is only non-zero for a very small x which is below the frequency range that we are concerned with. We then have

$$y_{\mathrm{NR}}\frac{\partial}{\partial x}(x^4(n' + n)) = 4nx^2 \ , \tag{7.64}$$

where the n^2 term, which is related to stimulated processes, is dropped because n is usually a very small number.

For $x \gg 1$, we may further drop nx^2, when compared with terms with x^4 in the partial derivative, and have $n' + n \approx 0$. It gives the solution that

$$n \propto e^{-x} \ , \tag{7.65}$$

which is the Wien's law: the intensity I_ν is related to the occupation number $n(\nu)$ as

$$I_\nu = \frac{2h\nu^3}{c^2} n(\nu) = \frac{2h\nu^3}{c^2} e^{-\frac{h\nu}{kT}} \ . \tag{7.66}$$

For $x \ll 1$, n can usually be neglected, compared with n'. Then the solution is

$$n \propto x^m \ , \tag{7.67}$$

where

$$m = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + \frac{4}{y_{\rm NR}}} \ . \tag{7.68}$$

The intensity is then

$$I_\nu \propto \nu^{3+m} \ . \tag{7.69}$$

The choice of the plus and minus signs in determining m is not trivial. For $y_{\rm NR} \ll 1$, the square-root term dominates. The minus sign is appropriate to avoid a positive m so that in such an optically thin case there is a simple decreasing power law to join the exponential cut off at high frequency. For $y_{\rm NR} \sim 1$ or larger, a linear combination of two power laws with different power indices should be considered. In general, when the media are more and more opaque, a thermal bump around $x \sim 1$ will appear in I_ν.

Chapter 8

Some Plasma Effects

In this chapter we discuss some phenomena related to the existence of a plasma. For simplicity, the plasma considered here is a system consisting of ions and electrons. Because ions are much more massive than electrons, they are considered to be at rest all the time, immersed in a sea of mobile electrons.

8.1 Dispersion measure

The first to discuss is the dispersion measure, which is used in estimating distance of radio pulsars. We will introduce the conductivity, dielectric constant, and plasma frequency of a plasma in the following and then discuss the dispersion measure.

The conductivity

Taking the variation of all quantities as $\exp(i(\vec{k}\cdot\vec{r}-\omega t))$, Maxwell's equations turn to be

$$\begin{aligned} i\vec{k}\cdot\vec{E} &= 4\pi\rho \\ i\vec{k}\times\vec{E} &= i\frac{\omega}{c}\vec{B} \\ i\vec{k}\cdot\vec{B} &= 0 \\ i\vec{k}\times\vec{B} &= \frac{4\pi}{c}\vec{j}-i\frac{\omega}{c}\vec{E} \end{aligned} \tag{8.1}$$

Consider electrons' equation of motion in response to the travelling electromagnetic waves, $m\dot{\vec{v}} = -e\vec{E}$ and then $\vec{v} = \frac{e\vec{E}}{i\omega m}$. The electric current density is then

$$\begin{aligned}
\vec{j} &= -n_{\rm e} e\vec{v} \\
&= \frac{in_{\rm e}e^2}{\omega m}\vec{E} \\
&= \sigma\vec{E}
\end{aligned} \tag{8.2}$$

where $n_{\rm e}$ is the electron number density and the conductivity is

$$\sigma \equiv \frac{in_{\rm e}e^2}{\omega m} \ . \tag{8.3}$$

The dielectric constant

From the continuity equation, $-i\omega\rho + i\vec{k}\cdot\vec{j} = 0$, we have

$$\rho = \frac{\vec{k}\cdot\vec{j}}{\omega} = \frac{\sigma}{\omega}\vec{k}\cdot\vec{E} \ . \tag{8.4}$$

With the defintion of the dielectric constant

$$\epsilon \equiv 1 - \frac{4\pi\sigma}{i\omega} \ , \tag{8.5}$$

Maxwell's equations become

$$\begin{aligned}
i\vec{k}\cdot\epsilon\vec{E} &= 0 \\
i\vec{k}\times\vec{E} &= i\frac{\omega}{c}\vec{B} \\
i\vec{k}\cdot\vec{B} &= 0 \\
i\vec{k}\times\vec{B} &= -i\frac{\omega}{c}\epsilon\vec{E} \ ,
\end{aligned} \tag{8.6}$$

which are 'source free'. From the above equations (Faraday's law and Ampere's law) we also have the dispersion relation, $\omega(\vec{k})$, to be

$$\epsilon\omega^2 = c^2k^2 \ . \tag{8.7}$$

The plasma frequency

The dielectric constant can be written as

$$\epsilon = 1 - \frac{\omega_{\rm p}^2}{\omega^2} \ , \tag{8.8}$$

where $\omega_{\rm p}$ is the plasma frequency and

$$\omega_{\rm p}^2 = \frac{4\pi n_{\rm e} e^2}{m} \ . \tag{8.9}$$

Now the dispersion relation, $\omega(\vec{k})$, becomes

$$\omega^2 = \omega_{\rm p}^2 + k^2 c^2 \ . \tag{8.10}$$

For $\omega < \omega_{\rm p}$, the wave number k is imaginary and no waves can propagate. For reference, the value of the plasma frequency is

$$\omega_{\rm p} = 5.63 \times 10^4 \sqrt{n_{\rm e}} \ {\rm s}^{-1} \ , \tag{8.11}$$

or,

$$\nu_{\rm p} = 0.9 \times 10^4 \sqrt{n_{\rm e}} \ {\rm Hz} \ , \tag{8.12}$$

where $n_{\rm e}$ is in Gaussian units.

The dispersion measure

The phase velocity of a photon is

$$v_\phi := \frac{\omega}{k} =: \frac{c}{n_{\rm r}} \ , \tag{8.13}$$

where the refraction index $n_{\rm r}$ is

$$n_{\rm r} = \sqrt{\epsilon} = \sqrt{1 - \frac{\omega_{\rm p}^2}{\omega^2}} < 1 \ . \tag{8.14}$$

Therefore we have $v_\phi > c$. The group velocity is

$$v_{\rm g} := \frac{\partial \omega}{\partial k} = c\sqrt{1 - \frac{\omega_{\rm p}^2}{\omega^2}} \ , \tag{8.15}$$

and therefore $v_{\rm g} < c$. Since the group velocity is a function of frequency, a pulse will be dispersed along the way of its propagation.

The time interval, T, for a photon to travel a distance d in a plasma is

$$T = \int_0^d \frac{\mathrm{d}s}{v_\mathrm{g}} \ . \tag{8.16}$$

For cases of $\omega \gg \omega_\mathrm{p}$, one can have $v_\mathrm{g} \approx c(1 - \frac{1}{2}\frac{\omega_\mathrm{p}^2}{\omega^2})$, and $\frac{1}{v_\mathrm{g}} \approx \frac{1}{c}(1 + \frac{1}{2}\frac{\omega_\mathrm{p}^2}{\omega^2})$. Then,

$$T \approx \frac{d}{c} + \frac{1}{2c\omega^2}\int_0^d \omega_\mathrm{p}^2 \mathrm{d}s \ . \tag{8.17}$$

For the application to radio pulsars, usually one can measure the arrival time difference for different frequencies, that is,

$$\frac{\mathrm{d}T}{\mathrm{d}\omega} = -\frac{4\pi e^2}{cm\omega^3}\mathcal{D} \ , \tag{8.18}$$

where

$$\mathcal{D} = \int_0^d n_\mathrm{e} \mathrm{d}s \tag{8.19}$$

is the dispersion measure. Given a model of n_e, the distance to the pulsar can be estimated.

8.2 Faraday rotation

The group velocity of electromagnetic waves in a plasma is frequency dependent. In a magnetized plasma, it is also polarization dependent. This is what we will discuss in this section.

Group velocity for different polarizations

If the plasma is not isotropic, e.g., there exists an external magnetic field, the dielectric constant is no longer a scalar. Furthermore, only waves with certain polarization (eigen modes) have the simple exponential form employed in the last section. Now let's consider a circularly polarized wave propagating along the direction of a constant external magnetic field, whose strength is much stronger than the wave field. Such a wave is an eigen mode in this medium. The equation of motion for electrons in the plasma is

$$m\dot{\vec{v}} = -e\vec{E} - \frac{e}{c}\vec{v} \times \vec{B} \ , \tag{8.20}$$

where

$$\vec{E}(t) = E_0 \exp(-i\omega t)(\hat{e}_1 \mp i\hat{e}_2) \tag{8.21}$$

is the wave field and

$$\vec{B} = B_0 \hat{e}_3 \tag{8.22}$$

is the external field. Here the '$-$' sign corresponds to the right-handed circular polarization. From the above equations one can find the velocity is related to the wave electric field as

$$\vec{v}(t) = \frac{-ie}{m(\omega \pm \omega_0)} \vec{E}(t) \ . \tag{8.23}$$

Comparison of this equation with previous ones in the last section (Eq.(8.2 and Eq.(8.5)) leads to

$$\epsilon_{\mathrm{R,L}} = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega(\omega \pm \omega_0)} \tag{8.24}$$

where R,L corresponds to the $+$ and $-$ signs in the denominator of the 2nd term at the right hand side respectively. The dispersion relation is still in the same form as that in Eq.(8.7), but we can see that these waves travel with different velocities.

The rotation measure

A linearly polarized wave is a *coherent superposition* of a right-hand and a left-hand circularly polarized wave. Its polarization plane will rotate along its propagation because of the different velocity of the two component waves. This effect is called **Faraday rotation**.

The phase angle that the electric-field vector of a circularly polarized wave will sweep through along traveling over a distance d is

$$\phi_{\mathrm{R,L}} = \int_0^d k_{\mathrm{R,L}} \mathrm{d}s \ , \tag{8.25}$$

where

$$\begin{aligned} k_{\mathrm{R,L}} &= \frac{\omega}{c}\sqrt{\epsilon_{\mathrm{R,L}}} \\ &\approx \frac{\omega}{c}\left(1 - \frac{\omega_{\mathrm{p}}^2}{2\omega^2}(1 \mp \frac{\omega_0}{\omega})\right) \ , \end{aligned} \tag{8.26}$$

where we have assumed that $\omega \gg \omega_{\rm p}$ and $\omega \gg \omega_0$. The polarization plane of the linearly polarized wave will be rotated by an angle $\Delta\theta$ equal to one half of the difference between $\phi_{\rm R}$ and $\phi_{\rm L}$. Therefore,

$$\begin{aligned}\Delta\theta &= \frac{1}{2}\int_0^d (k_{\rm R} - k_{\rm L}){\rm d}s \\ &= \frac{1}{2}\int_0^d \frac{\omega_0}{c}\frac{\omega_{\rm p}^2}{\omega^2}{\rm d}s \\ &= \frac{2\pi e^3}{m^2c^2\omega^2}\int_0^d n_{\rm e}B_{\|}{\rm d}s \end{aligned} \tag{8.27}$$

The $B_{\|}$ is the component along the line of sight. Although we only consider here a simplified case that $\vec{B}$ is itself along the line of sight, it can be shown that this result is in general correct. One may measure the polarization angle at different frequencies, i.e., $\frac{{\rm d}\Delta\theta}{{\rm d}\omega}$, to infer the value of the integral $\int n_{\rm e}B_{\|}{\rm d}s$, which is sometimes called the **rotation measure** and can give some information about the field strength if $n_{\rm e}$ and d are provided.

8.3 Cherenkov radiation and Razin effect

The radiation emitted by charged particles is subject to all the plasma propagation effects. Radiation of frequency lower than the plasma frequency cannot propagate. A pulse will disperse because the group velocity is frequency dependent. The polarization angle of a linearly polarized light will rotate due to Faraday rotation, and the polarization of synchrotron radiation will be degraded because Faraday rotation is frequency dependent. In the following we will describe two more effects, which involve induced motions and emissions of the particles comprising the medium. Our description here is only a simplified, qualitative one.

When charged particles move at a speed larger than the phase speed of light in the medium, i.e., $v > c/n_{\rm r}$, they emit **Cherenkov radiation**. Here we should consider only the case of $n_{\rm r} > 1$ and we note that $n_{\rm r}$ is in general frequency dependent. Discussion of the Cherenkov radiation can be found in Jackson (1975, page 639) and Longair (2011, page 264). Roughly speaking, wavefronts of the electromagnetic field variation caused by the moving charge are coherently summed in the direction of $\cos\theta = \frac{c}{n_{\rm r}v}$, where θ is the angle between the charge motion and the radiation. The spectrum of Cherenkov radiation is, in term of specific intensity,

$$I_\omega(\omega) \propto \omega(1 - \frac{c^2}{n_\mathrm{r}^2 v^2}) \ . \tag{8.28}$$

The refractive index n_r is usually a complicated function of ω. The example of water as the medium is shown in Figure 8.1. For cosmic-ray or gamma-

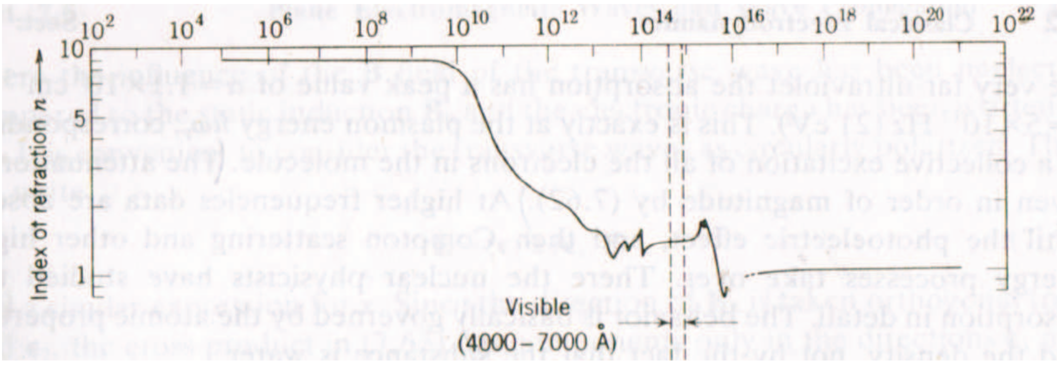

Figure 8.1: The refraction index of water. The abscissa labeled at the top is the photon frequency. This figure is taken from Jackson (1975, page 291, Figure 7.9).

ray induced pair cascade in the upper atmosphere, very often the blue light dominates in their Cherenkov radiation. When there is charge asymmetry in the secondary charged particles, coherent Cherenkov radiation from a bunch of charges can be emitted at longer wavelengths. This is the **Askaryan effect**, usually observed in microwave bands (Saltzberg et al. 2001).

Now let's discuss the Razin effect. If $n_\mathrm{r} < 1$, as it is in a cold plasma, the beaming effect will be suppressed at lower frequencies. This can be seen by replacing the speed of light in vacuum with the phase speed of light in a medium. In our earlier discussion, we have the beaming angle θ_b given by $\theta_b \sim 1/\gamma = \sqrt{1-\beta^2}$ in a vacuum. Now, in a medium, we should have

$$\theta_b \sim \sqrt{1 - n_\mathrm{r}^2 \beta^2} \ . \tag{8.29}$$

If n_r is very close to unity, θ_b is determined by β. On the other hand, if n_r is substantially smaller than unity, we will have

$$\theta_b \sim \sqrt{1 - n_\mathrm{r}^2} = \frac{\omega_\mathrm{p}}{\omega} \ . \tag{8.30}$$

For frequencies of

$$\omega < \gamma\omega_{\mathrm{p}} \ , \tag{8.31}$$

we have $\theta_b > 1/\gamma$ and the medium effect is important. In this frequency regime, synchrotron (and curvature) radiation will be cut off because of beaming suppression. This is called the **Razin effect**. Considering $\omega_{\mathrm{c}} \approx \gamma^2\omega_0$, we see that the synchrotron radiation of charges with $\gamma < \omega_{\mathrm{p}}/\omega_0$ will be largely suppressed.

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