VERY
A Simple Test for Diffusion-~~Loss~~ Equation

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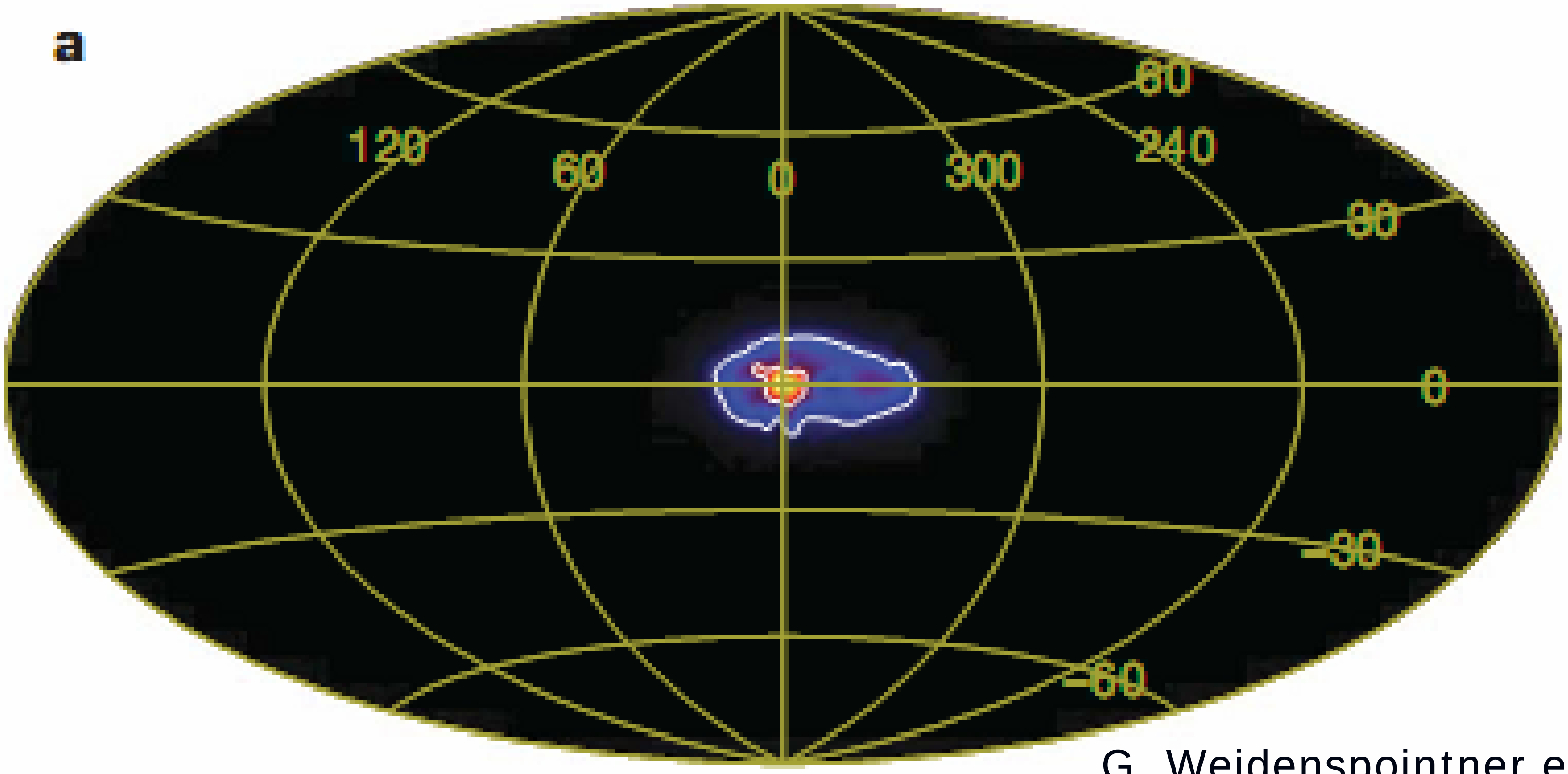
Outline

- Introduction
- Diffusion-Loss Equation
- Numerical Method
- Simple Test
- Future Work

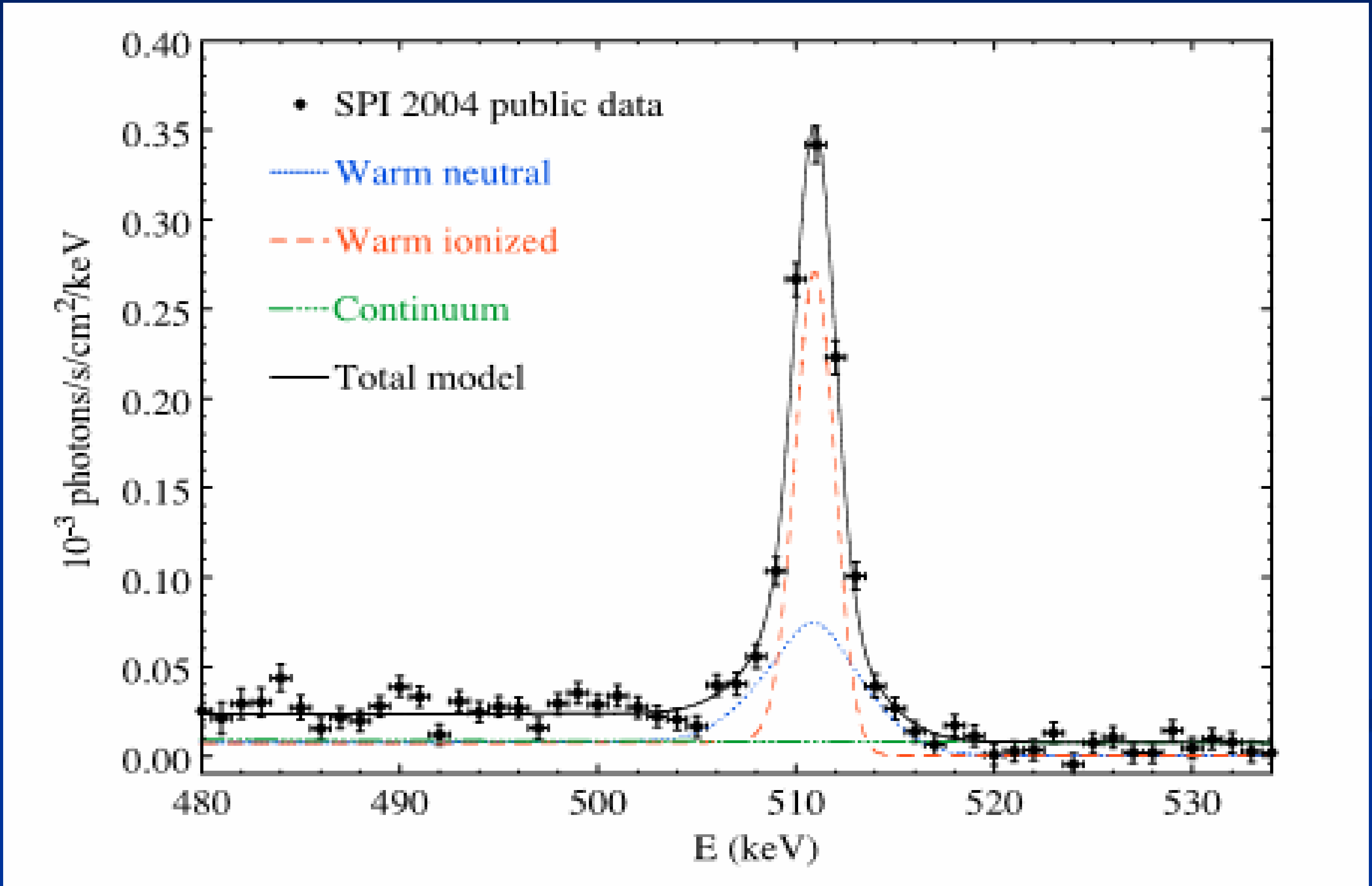
Introduction



G. Weidenspointner et al. 2008 Nature

Introduction

Jean et al. 2006 A&A



Introduction

- The bulge emission is spherically symmetric and is centered on the galactic center with an extension of ~ 5^0-8^0 (*FWHM*)
- Total flux= $\left(1.09 \pm 0.04\right) \times 10^{-3}\ photons / s \cdot cm^2$

Board line : $(0.35 \pm 0.11) \times 10^{-3}\ photons / s \cdot cm^2$

Narrow line : $(0.72 \pm 0.12) \times 10^{-3}\ photons / s \cdot cm^2$ (Jean et al. 2006 A&A)

Introduction

- Line width ~ $2.37 \pm 0.25 keV$ (Churazov et al., 2005)

Board line : $(5.4 \pm 1.2) keV$

Narrow line : $(1.3 \pm 0.4) keV$ (Jean et al., 2006)

- Positronium fraction ~ 0.967 ± 0.022

(Jean et al., 2006)

- production positron rate ~ $10^{43} s^{-1}$ (J. Knodlseder et al., 2005)
- $\frac{L_B}{L_D} \sim 1.7 \pm 0.4$ (J. C. Higdon et al. astro-ph/0711.3008)

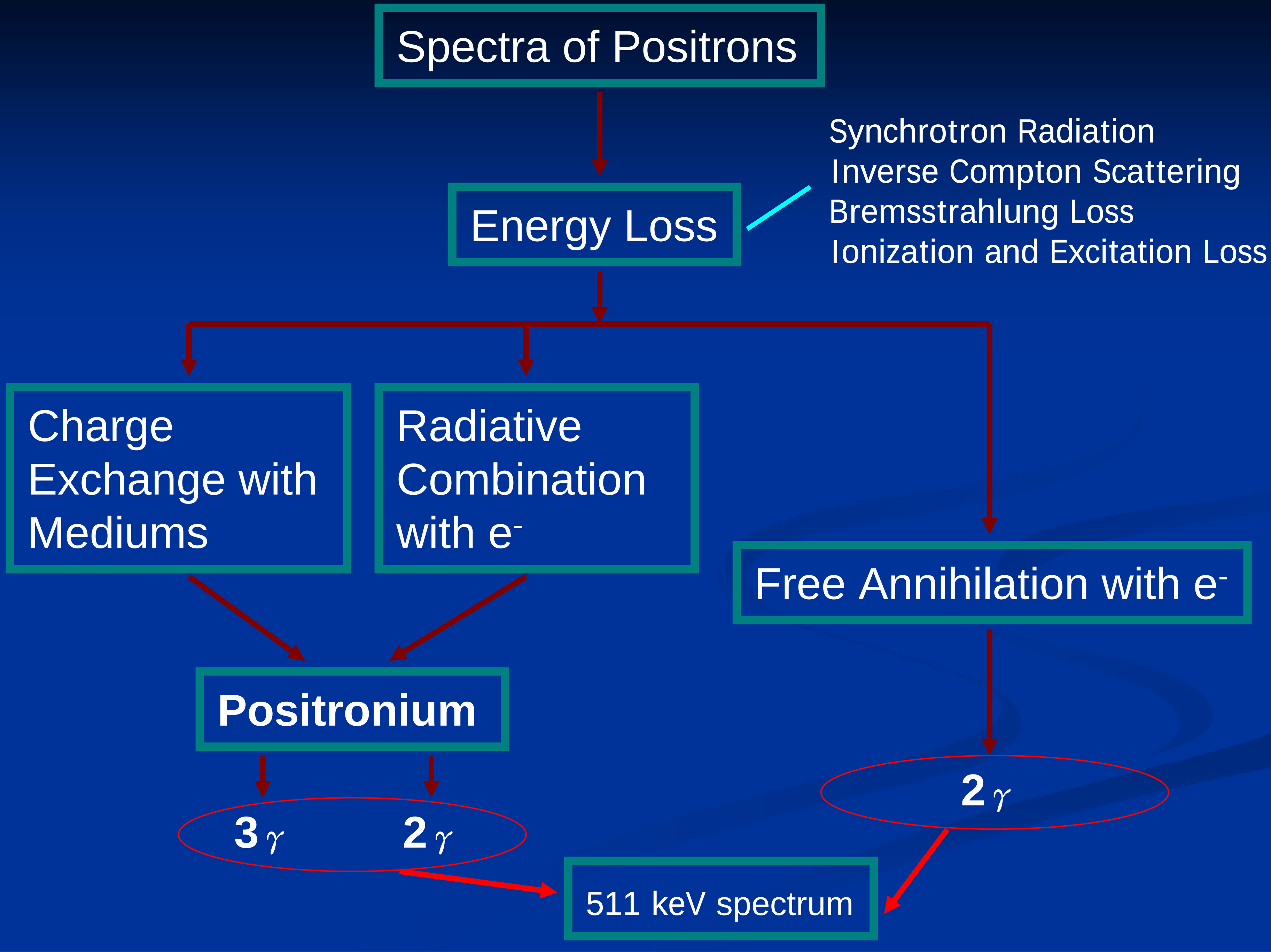
Introduction

- Most the propagation of cosmic rays is considered in the system of the diffusion model.
- Diffusion-loss equation:

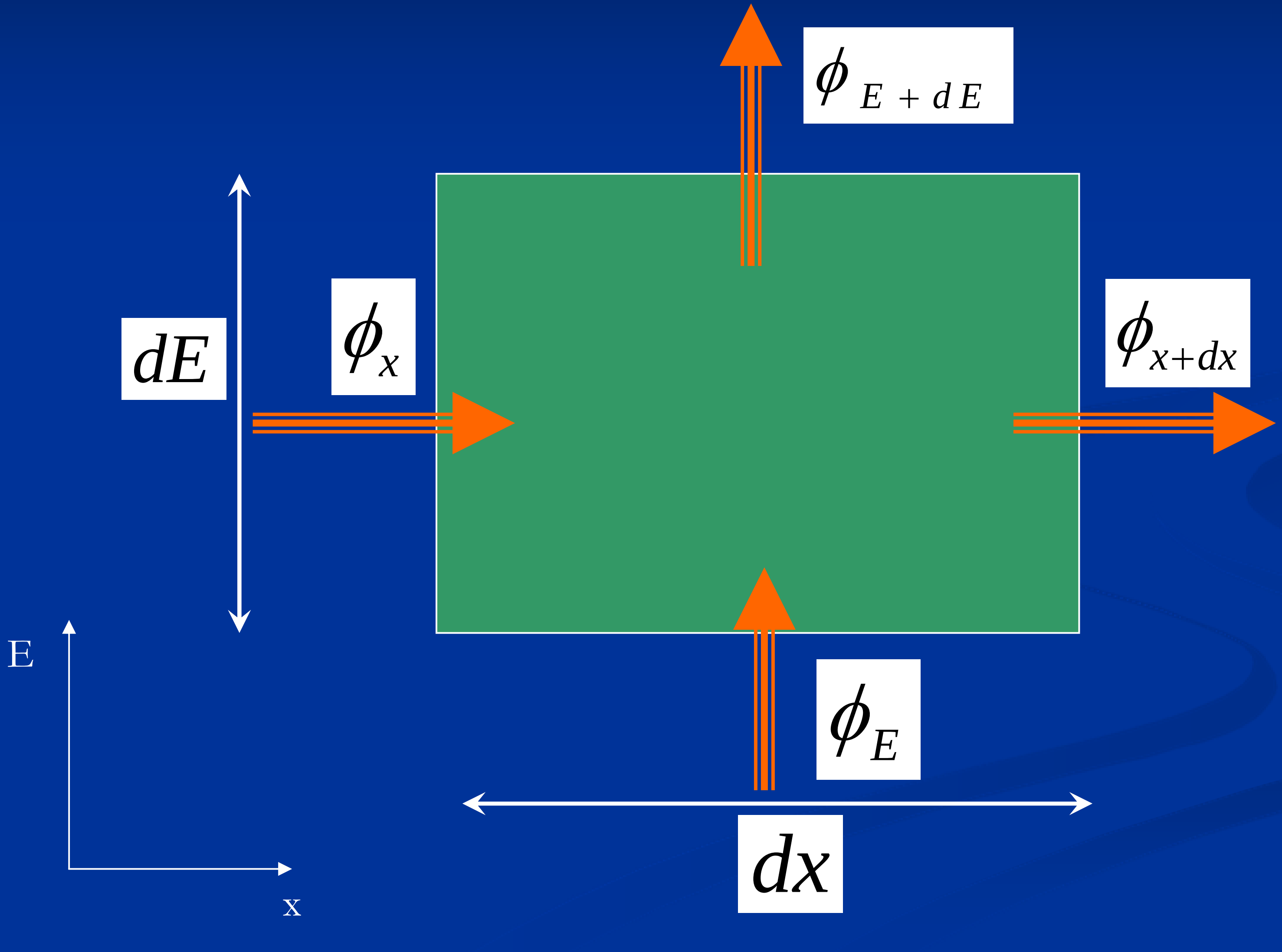
$$\frac{\partial N}{\partial t}-\nabla\bullet\left(\widehat{D}\nabla N\right)+\frac{\partial}{\partial E_k}\left(\frac{dE_k}{dt}N\right)+\frac{N}{\tau}=Q\left(\vec{r},E_k,t\right)$$

$$N=N\left(\vec{r},t,E_k\right)$$

$$Q\left(\vec{r},E_k,t\right)=A\left(E_k\right)T\left(t\right)X\left(\vec{r}\right)$$


Spectra of Positrons
Energy Loss
Synchrotron Radiation
Inverse Compton Scattering
Bremsstrahlung Loss
Ionization and Excitation Loss
Charge Exchange with Mediums
Radiative Combination with e-
Free Annihilation with e-
Positronium
3 γ
2 γ
2 γ
511 keV spectrum

Diffusion-Loss Equation



Diffusion-Loss Equation

$$\frac{\partial N\left(x,E,t\right)}{\partial t}dEdx=$$

$$\left[\phi_x\left(x,E,t\right)-\phi_{x+dx}\left(x+dx,E,t\right)\right]dE$$

$$+\left[\phi_E\left(x,E,t\right)-\phi_{E+dE}\left(x,E+dE,t\right)\right]dx$$

$$\frac{\partial N}{\partial t}=-\frac{\partial \phi_x}{\partial x}-\frac{\partial \phi_E}{\partial E}$$

Diffusion-Loss Equation

$$\phi_x = -D\frac{\partial N}{\partial x}$$

$$N(E)\frac{dE}{dt} = \phi_E = -b(E)N(E)$$

$$\frac{\partial N}{\partial t} - \nabla\bullet\left(\hat{D}\nabla N\right) + \frac{\partial}{\partial E}\left[bN\right] = 0$$

$$\frac{\partial N}{\partial t} - \nabla\bullet\left(\widehat{D}\nabla N\right) + \frac{\partial}{\partial E_k}\left(\frac{dE_k}{dt}N\right) + \frac{N}{\tau} = Q\left(\vec{r}, E_k, t\right)$$

Numerical Method

- Finite difference method with the explicit scheme $N_{i,j}^{n}$
- Consider spherical symmetric case, and D=constant

$$N_{i,j}^{n+1} =$$

$$N_{i,j}^{n} + \frac{D\Delta t}{(\Delta r)^2}\left(N_{i+1,j}^{n} + N_{i-1,j}^{n} - 2N_{i,j}^{n}\right)$$

$$+ \frac{D\Delta t}{r_i(\Delta r)}\left(N_{i+1,j}^{n} + N_{i-1,j}^{n}\right)$$

$$- \frac{\Delta t}{\Delta E}\left(b_j N_{i,j}^{n} + b_{j-1} N_{i,j-1}^{n}\right) - \frac{\Delta t}{\tau} N_{i,j}^{n} + Q_{i,j}^{n}\Delta t$$

Numerical Method

- Considering a simple case first, we drop out the energy dependence, therefore

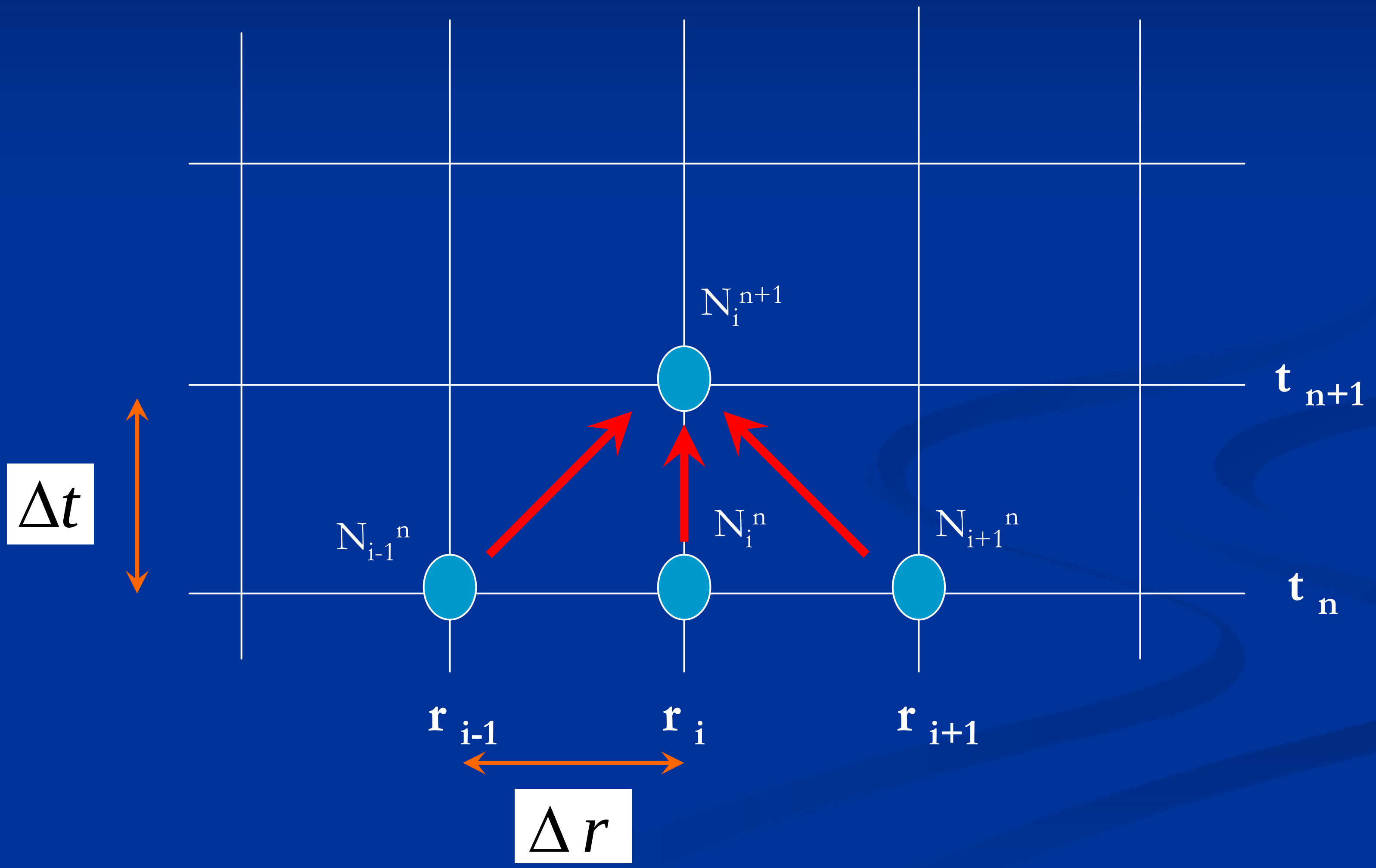
$$N_i^n$$

$$\frac{\partial N}{\partial t} - \frac{D}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial N}{\partial r} \right) = Q(r,t)$$

$$N_{i,j}^{n+1} = N_{i,j}^{n} + \frac{D\Delta t}{(\Delta r)^2}\left(N_{i+1,j}^{n} + N_{i-1,j}^{n} - 2N_{i,j}^{n} \right)$$
$$+ \frac{D\Delta t}{r_i(\Delta r)}\left(N_{i+1,j}^{n} + N_{i-1,j}^{n} \right) + Q_{i,j}^{n}\Delta t$$

$$\Leftrightarrow \quad \underset{\sim}{N}^{n+1} = \underset{\sim}{M}\,\underset{\sim}{N}^{n} + \underset{\sim}{G}$$

Numerical Method



Numerical Method

- Stability analysis:

  For a stable difference scheme small errors in the initial conditions cause small errors in the solution

- von Neumann stability analysis:

$$N_a^n = \xi^n e^{ik(a\Delta r)}$$

k is a real spatial wave number and $\xi = \xi$ (k) is a complex number

Numerical Method

- The number ξ is called the amplification factor at a given wave number k

$$\xi = (1 - 2\alpha) + 2\alpha \cos k\Delta r + i\beta_i \sin k\Delta r$$

$$\alpha = \frac{D \Delta t}{(\Delta r)^2}$$

$$\beta_i = \frac{2 D \Delta t}{r_i (\Delta r)}$$

Stable criterion of this scheme:

$$\frac{\beta_i^2}{2} \le \alpha \le \frac{1}{2}$$

Numerical Method

- Crank-Nicolson scheme:

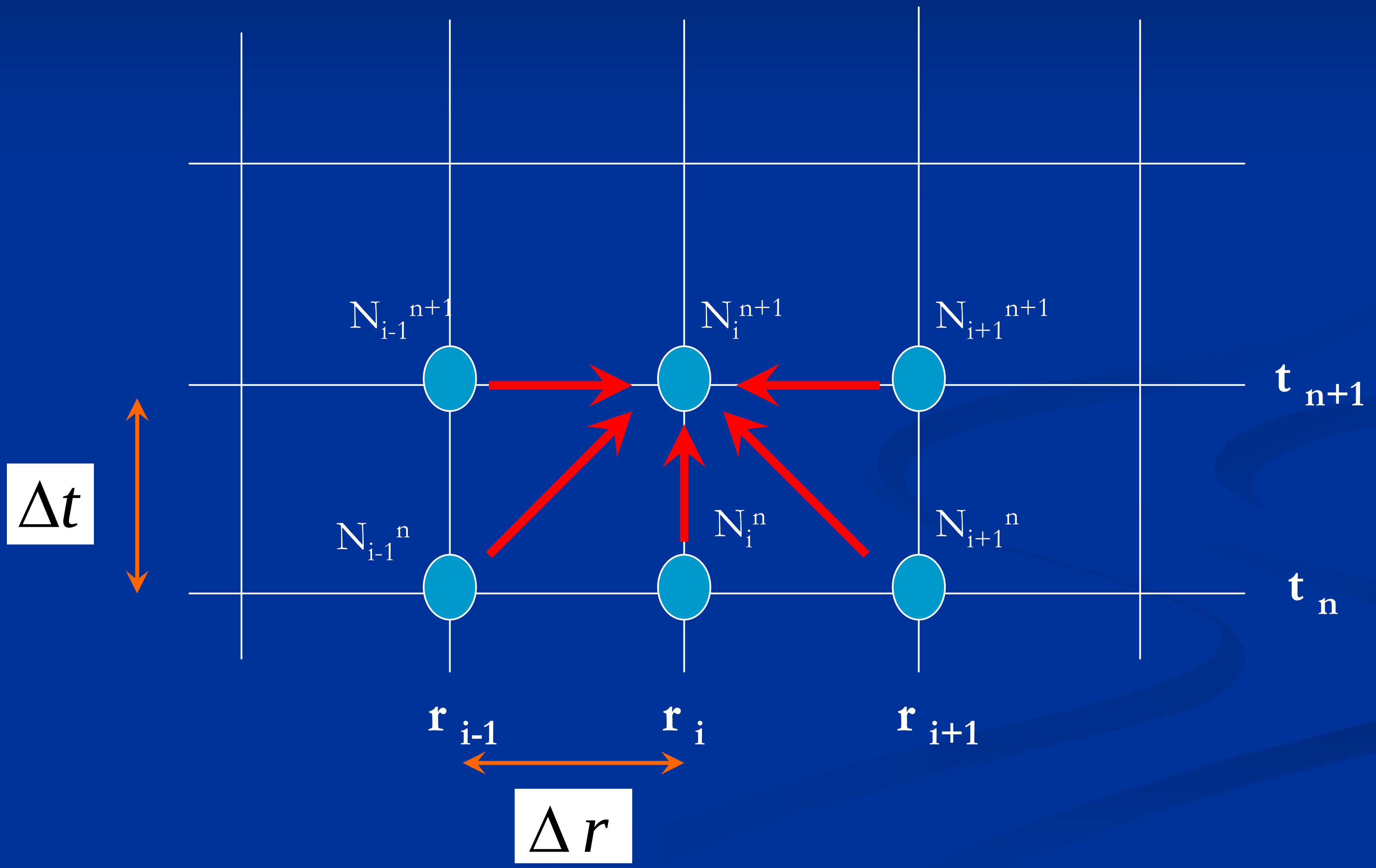
$$N_{i,j}^{n+1} =$$

$$N_{i,j}^{n} +$$

$$\frac{D\Delta t}{(\Delta r)^2}\left[\frac{1}{2}\left(N_{i+1,j}^{n+1} + N_{i-1,j}^{n+1} - 2N_{i,j}^{n+1}\right) + \frac{1}{2}\left(N_{i+1,j}^{n} + N_{i-1,j}^{n} - 2N_{i,j}^{n}\right)\right]$$

$$+\frac{D\Delta t}{r_i(\Delta r)}\left[\frac{1}{2}\left(N_{i+1,j}^{n+1} + N_{i-1,j}^{n+1}\right) + \frac{1}{2}\left(N_{i+1,j}^{n} + N_{i-1,j}^{n}\right)\right] + Q_{i,j}^{n}\Delta t$$

Numerical Method



Numerical Method

$$\xi = \frac{(1-\alpha)+\alpha\cos(k\Delta r)+\frac{i}{2}\beta_i\sin(k\Delta r)}{(1+\alpha)-\alpha\cos(k\Delta r)-\frac{i}{2}\beta_i\sin(k\Delta r)}$$

$$\rightarrow \quad |\xi|^2 = \frac{1-4\alpha\sin^2\left(\frac{k\Delta r}{2}\right)+4\alpha\sin^2\left(\frac{k\Delta r}{2}\right)+\frac{\beta_i^2}{4}\sin^2(k\Delta r)}{1+4\alpha\sin^2\left(\frac{k\Delta r}{2}\right)+4\alpha\sin^2\left(\frac{k\Delta r}{2}\right)+\frac{\beta_i^2}{4}\sin^2(k\Delta r)} \le 1$$

- We can find that this scheme is unconditionally stable for any size Δt

Simple Test

- Consider spherical symmetric:

$$\frac{\partial N}{\partial t} - \frac{D}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial N}{\partial r}\right) = Q(r,t)$$

- Initial condition:

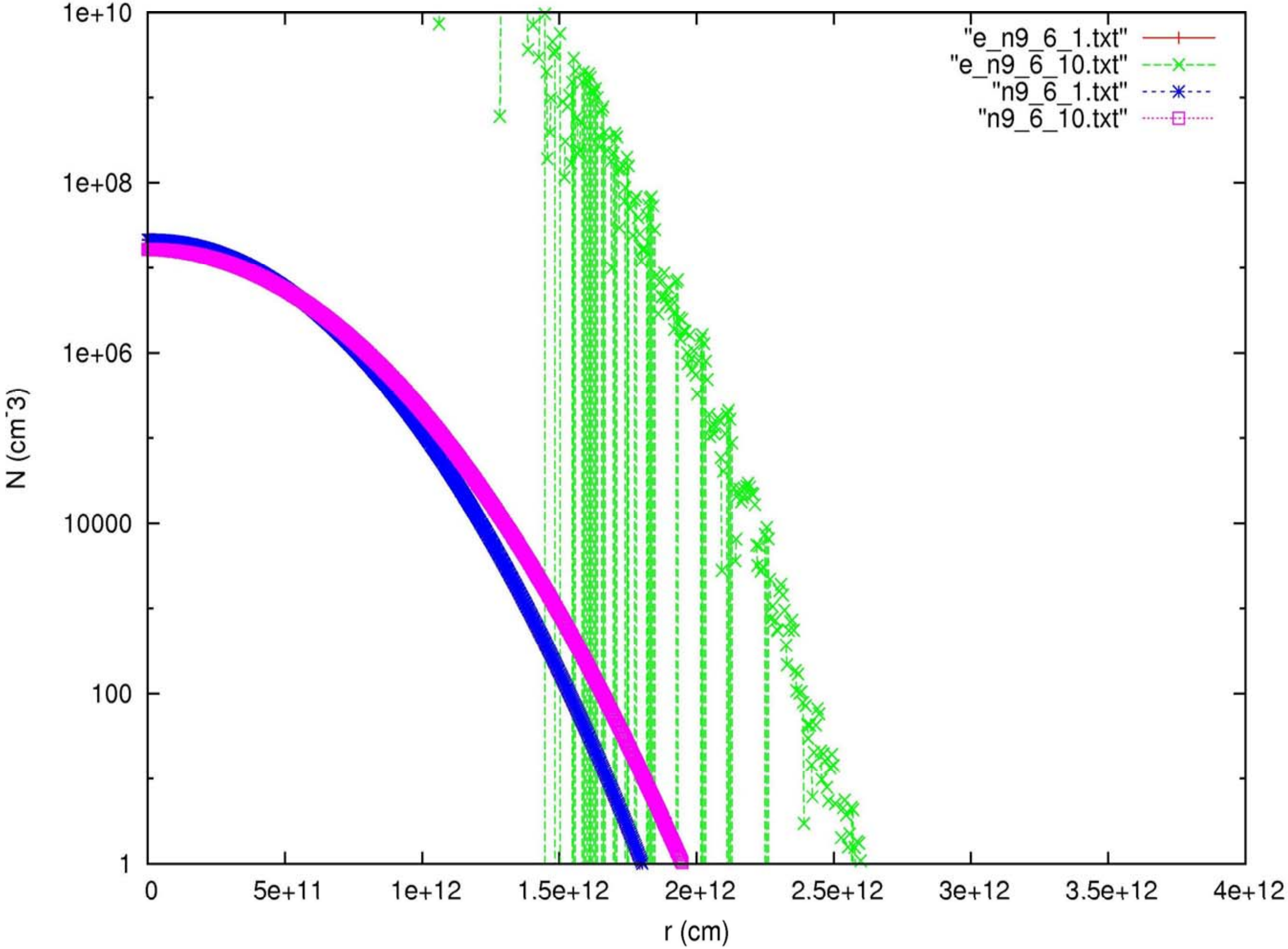
$$N(r, t=0) = 0$$

- Source:

$$Q = N_0 \times G(r, r_0 = 0, \sigma_r) \times \delta(t)$$

$$N_0 = 10^{43}$$

$$\sigma_r = 10^{-7}\,pc$$



"e_n9_6_1.txt"
"e_n9_6_10.txt"
"n9_6_1.txt"
"n9_6_10.txt"
1e+10
1e+08
1e+06
10000
100
1
N (cm^-3)
0
5e+11
1e+12
1.5e+12
2e+12
2.5e+12
3e+12
3.5e+12
4e+12
r (cm)

Future Work

- Construct the complete numerical program
- Consider more reality conditions

>>Thank You<<

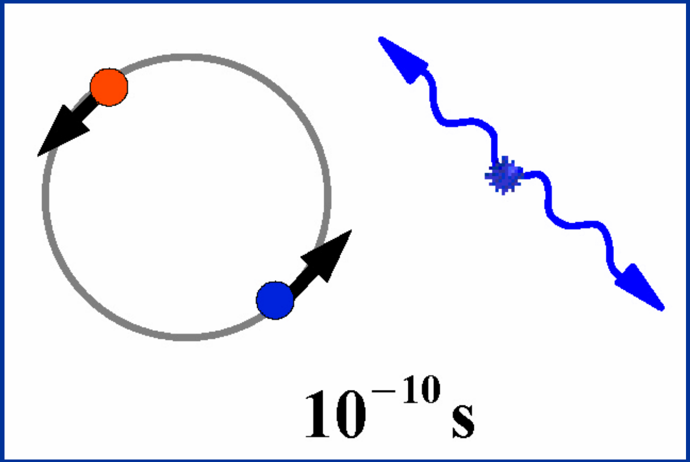
Introduction

- Problems:
- The production of positrons
- The galactic map of the annihilation line
- The propagation of the positrons between their production sites and annihilation places

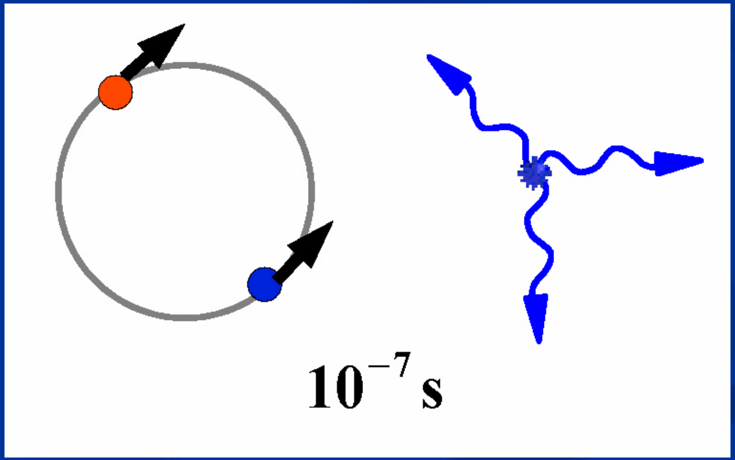
Introduction

- Positronium (PS):
- It is the bound state of e^{+} and e^{-}

Para-PS state



Ortho-PS state



Introduction

- The chemical composition of ISM :
- Hydrogen: 90.8%
- Helium: 9.1%
- Heavier elements: 0.12%

(K. M. Ferriere, 2001)

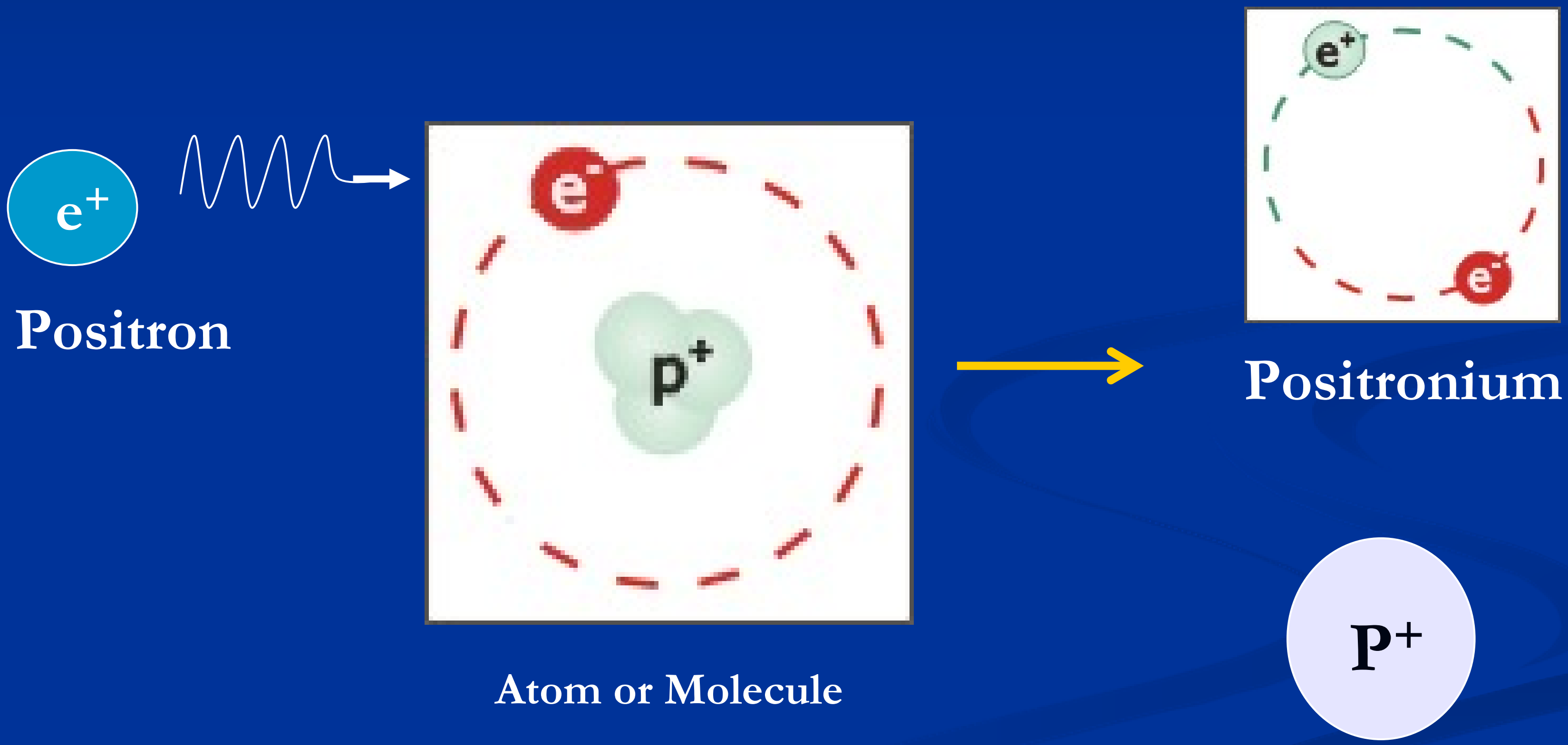
Introduction

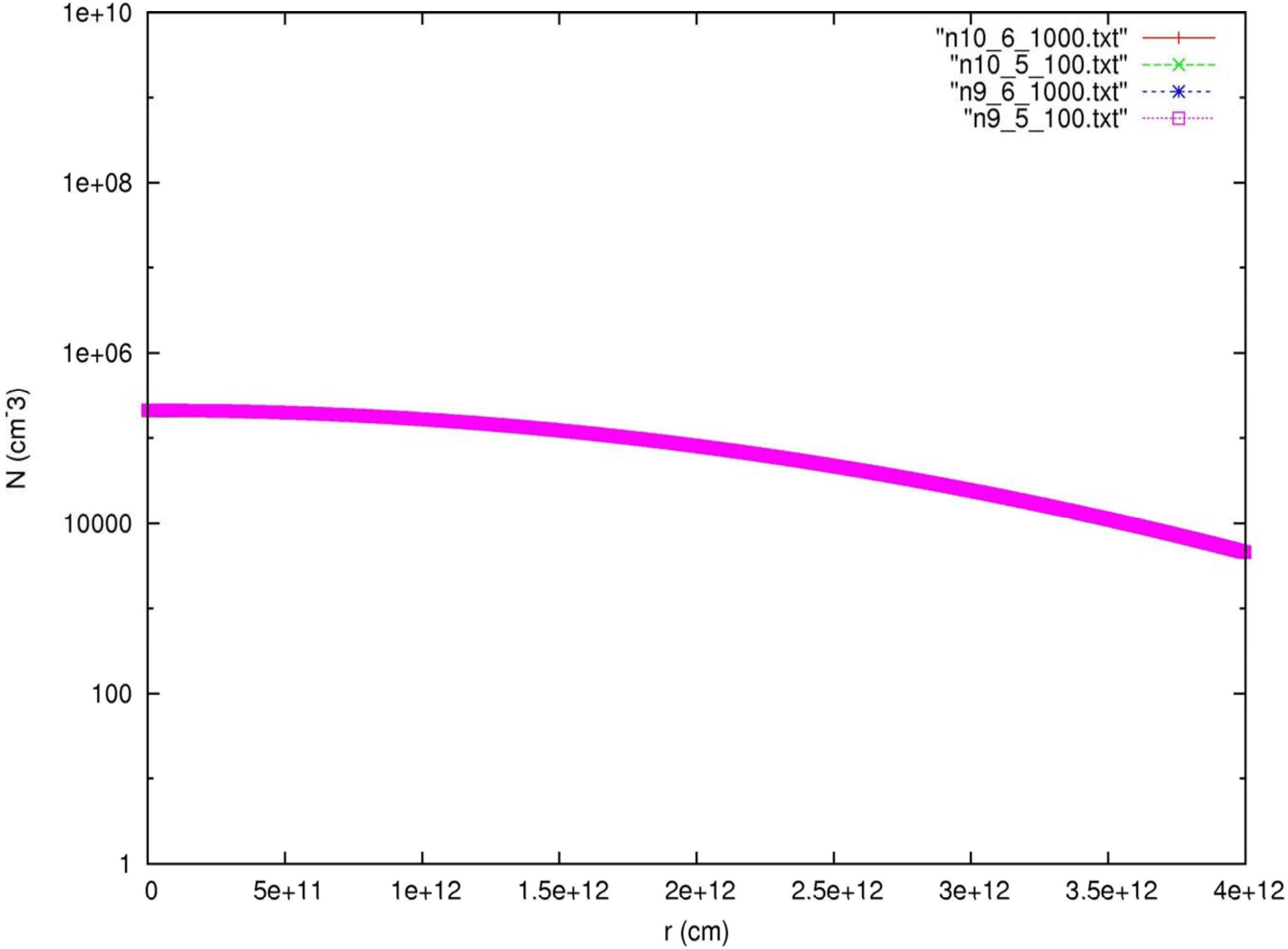
- ISM model (McKee & Ostriker,1977)
- 2.4% cold phase, n=42 cm^{-3}, T=80K
- 23% warm neutral phase, n=0.37 cm^{-3} , T=8000K
- 23% warm ionized phase, n=0.25 $cm^{-3,}$, T=8000K
- 52% hot phase, n=$3.5*10^{-3}$ cm^{-3} , T=$4.5*10^{5}$K

(N. Guessoum et al., 2005)

Introduction

- Charge Exchange (CE):





"n10_6_1000.txt"
"n10_5_100.txt"
"n9_6_1000.txt"
"n9_5_100.txt"
N (cm^-3)
r (cm)
1e+10
1e+08
1e+06
10000
100
1
0
5e+11
1e+12
1.5e+12
2e+12
2.5e+12
3e+12
3.5e+12
4e+12

Introduction

- Berezinkii et al. estimated the average value of the diffusion coefficient D → $\sim 10^{27}$ cm^{2}s^{-1}

(K.S. Cheng et al.,2006)

$$\frac{\partial N}{\partial t} - \nabla \bullet \left(\widehat{D} \nabla N \right) + \frac{\partial}{\partial E_k} \left(\frac{dE_k}{dt} N \right) + \frac{N}{\tau} = Q\left(\vec{r}, E_k, t \right)$$

Diffusion-Loss Equation

$$\frac{\partial N}{\partial t} = -\frac{\partial \phi_x}{\partial x} - \frac{\partial \phi_E}{\partial E} + Q$$

$$\phi_x = N\vec{v} - D\frac{\partial N}{\partial x}$$

$$N(E)\frac{dE}{dt} = \phi_E = -b(E)N(E)$$

$$\frac{\partial N}{\partial t} + \nabla \bullet (N\vec{v}) - \nabla \bullet (\hat{D}\nabla N) + \frac{\partial}{\partial E}[bN] = Q$$

$$\frac{\partial N}{\partial t} - \nabla \bullet (\widehat{D}\nabla N) + \frac{\partial}{\partial E_k}\left(\frac{dE_k}{dt}N\right) + \frac{N}{\tau} = Q(\vec{r}, E_k, t)$$

Introduction

G. Weidenspointner et al. astro-ph/0702621

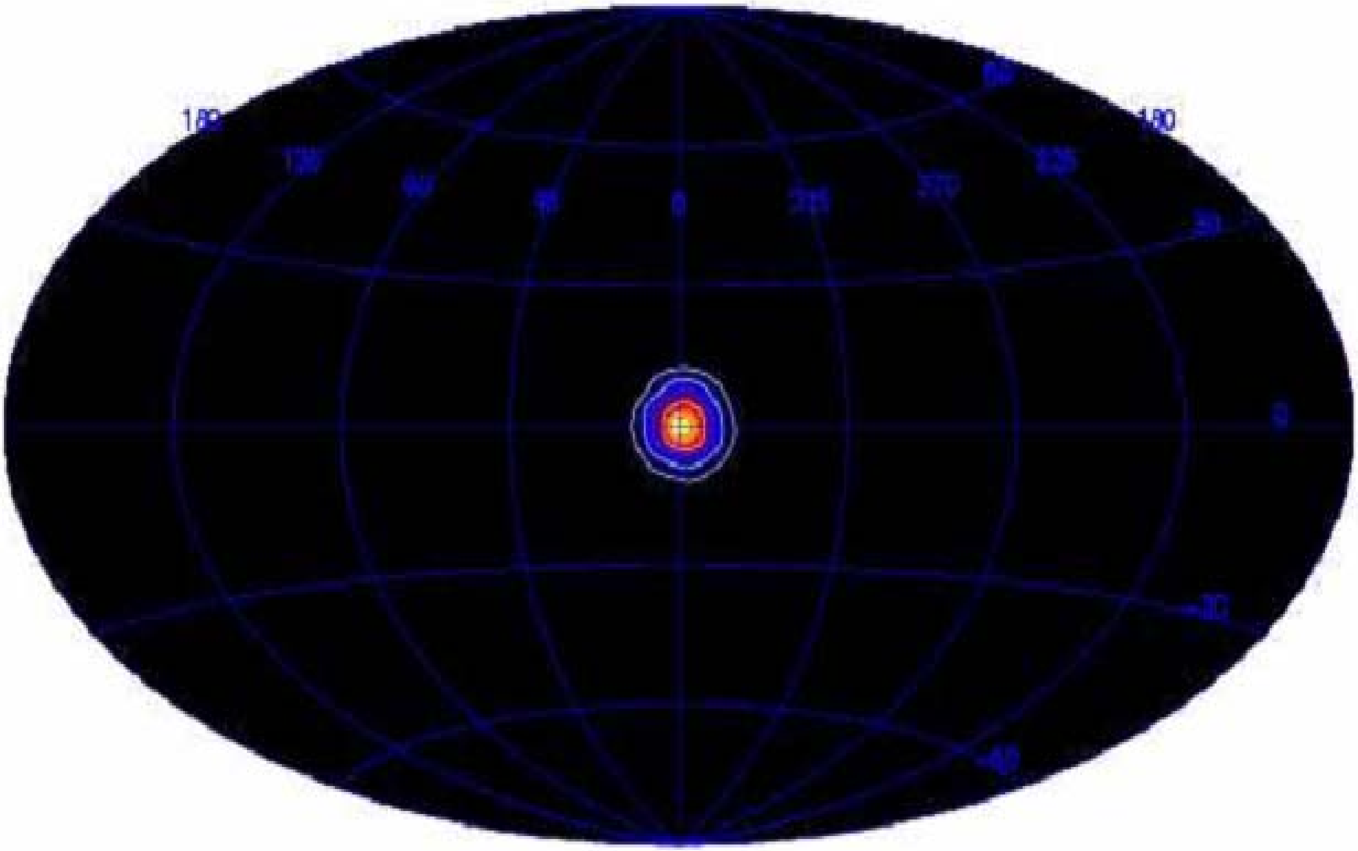
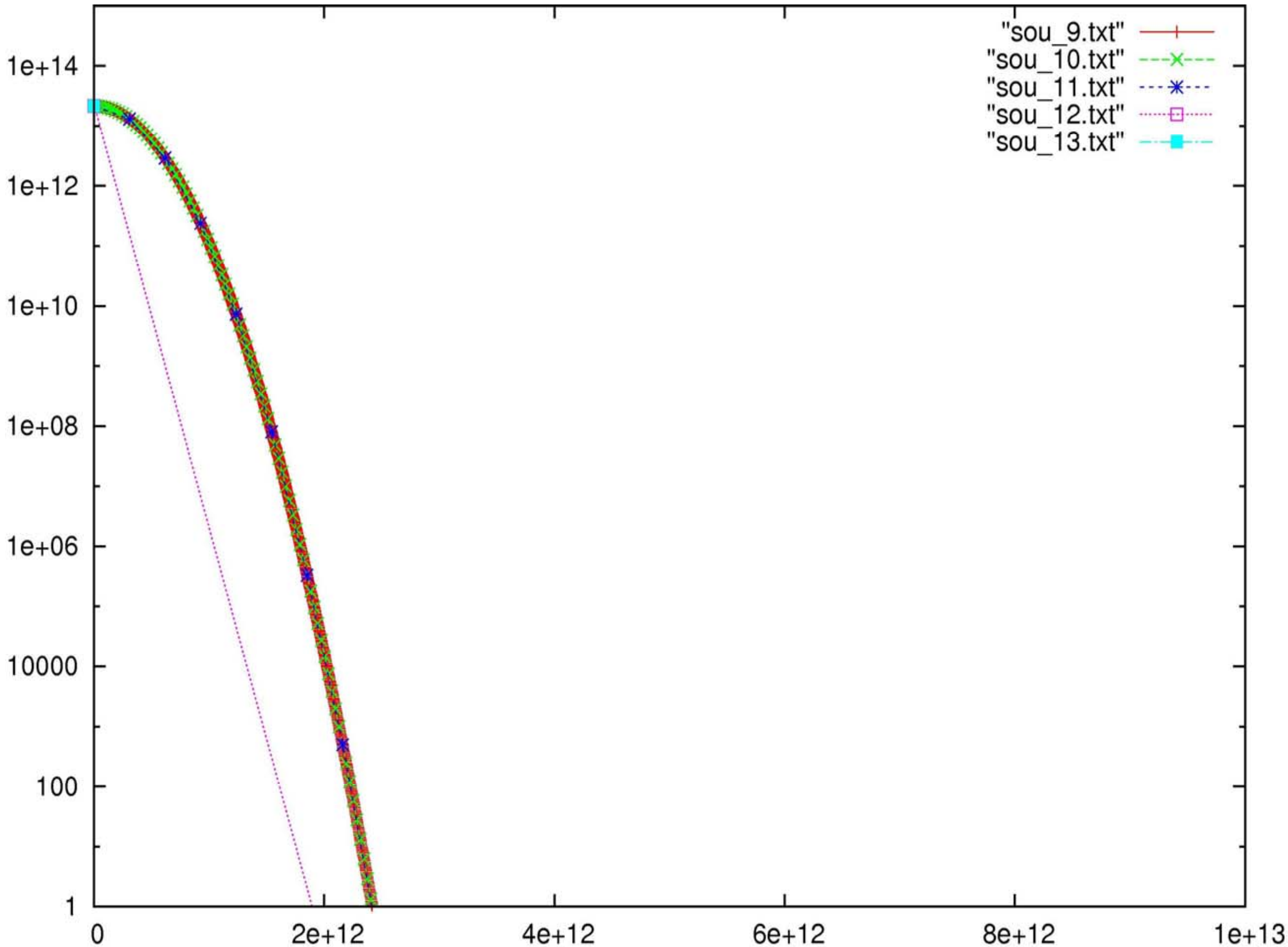


Figure 2. An MREM sky map of the 511 keV positron annihilation line emission. The contours indicate intensity levels of 10^{-2}, 10^{-3}, and 10^{-4} ph cm^{-2} s^{-1} sr^{-1}. Details are given in the text.



"sou_9.txt"
"sou_10.txt"
"sou_11.txt"
"sou_12.txt"
"sou_13.txt"
1e+14
1e+12
1e+10
1e+08
1e+06
10000
100
1
0
2e+12
4e+12
6e+12
8e+12
1e+13