From space-time to gravitation waves

Bubu
2008 Oct. 24
Do you know what the hardest thing in nature is? and that’s not diamond.

Space-time!
Because it’s almost impossible for you to change its structure.
Newton’s Gravity = Einstein’s space-time geometry

Wheeler: “matter tell space-time how to curve and space-time tell matter how to move”
• Gravitational radiation (wave) is the ripple _ space-time.
Outline

1. Introduction
2. How motions of matter affect the space-time and the idea of gravitational radiation/wave
3. Order of magnitude estimation
4. Conclusion and Prospect
The idea of gravitational wave:

Small-scale ripple propagate in a background of large-scale curvature
(Short Wave Approximation)
short wave limit:

\[ a \ll 1 \]

\[ \hat{\lambda} / R_{s-t} \ll 1 \]
constraint on sources:

\[ \frac{R_{source}}{\hat{\lambda}} \ll 1 \]

\[ \frac{M_{source}}{R_{source}} \ll 1 \]
How motions of matter affect the space-time?

- Matter with steady velocity
- Matter with acceleration
- Matter with rotation
- Two matters attract to each other via gravity
- Two matters in orbit with steady angular velocity
Radiation from a electric dipole source:

\[ \text{dipole} = d \equiv ex \]

\[ \langle \text{charge} \times \text{length} \rangle \]

\[ \text{Power} \propto \ddot{d}^2 = e^2 a^2 \]

and its gravitational radiation analog:

\[ \ddot{d} \leftrightarrow m_1 \ddot{a}_1 + m_2 \ddot{a}_2 = 0 \]
Radiation from a magnetic dipole source:

\[ \text{dipole} = J \equiv \sum r \times \rho \nu \]

\[ \text{Power } \propto \dot{J}^2 \]

and its gravitational radiation analog:

\[ \ddot{\mu} \leftrightarrow \frac{d^2}{dt^2} \left( \sum r \times m_i \nu_i \right) = 0 \]
GW emitted from two masses in orbit - Quadrupole radiation

\[ \text{quadrupole} = I^{mn} = \sum \rho x^m x^n \]

\[ \text{Power} \equiv L_{GW} \propto \ddot{I}^2 \]

Axial symmetry rotation
→ constant quadrupole moment
→ no gravitational waves
Order of magnitude estimation

\[ \text{quadrupole} = I \]
\[ \langle \text{mass} \times \text{length} \times \text{length} \rangle \]
\[ \text{Power} \equiv L_{GW} \propto \ddot{I}^2 \]

\[ \sim MR^2 \]
\[ \sim \left( \frac{MR^2}{T^3} \right)^2 \]
\[ \sim \left( \frac{\text{(mass of that part of system which moves)} \times \text{(size of system)}^2}{\text{(time for masses to move from one side of system to other)}^3} \right)^2 \]
\[ \sim (\text{energy/period})^2 \]
The maximum power of GW

Geometrized unit: $c = G = 1$

Virial theorem:
(describe the relation between the inertial energy and the gravitational energy)

\[
\frac{M}{R} \sim \nu^2
\]

\[
L_{GW} \sim \left(\frac{MR^2}{T^3}\right)^2 \sim M^2 R^4 \omega^6 \sim \left(\frac{M}{R}\right)^2 (R \omega)^6 \sim \left(\frac{M}{R}\right)^2 \nu^6 \leq \left(\frac{M}{R}\right)^5
\]

The maximum power output occurs when the system is near its gravitational radius and the formula breaks down

\[
L_{GW} \leq 1 = \frac{c^5}{G} = (3.63 \times 10^{59} \text{ erg/sec})
\]
GW emitted form a Rotating Bar

\[
L_{GW} \sim \left( \frac{MR^2}{T^3} \right)^2 \times (3.63 \times 10^{59} \text{ erg / sec})
\]

\[
\sim (Ml^2 \omega^3)^2 \times (3.63 \times 10^{59} \text{ erg / sec})
\]
GW emitted from the rotating of four masses at the corner of a square

\[ I^{xx} = 2\rho \times a^2 \]

No gravitational radiation
Price Theorem:

Whatever can be radiated is radiated
What’s their shape after Collapsing and becoming a static black hole?
Linear Theory

\[ ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu}dx^\mu dx^\nu \]

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad h_{\mu\nu} \ll 1 \]

\[ \bar{h}^{\mu
u} = h^{\mu
u} - \frac{1}{2} \eta^{\mu
u}h^\alpha_\alpha \]

\[ (-\frac{\partial^2}{\partial t^2} + \nabla^2)\bar{h}^{\mu
u} = 0 \quad \text{if in vacuum} \]

for GW propagates along z axis:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & h_+ & h_\times & 0 \\
0 & h_\times & -h_+ & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
t \\
x \\
y \\
z
\end{pmatrix}
\]

\[ h_\times = 0 \quad \text{(plus mode)} \]

\[ h_+ = 0 \quad \text{(cross mode)} \]

which changes the proper length between test particles
GW Polarization

- At any moment of time, a gravitational wave is invariant under a rotation of 180 degree → spin=2
- Shear tensor
- The two basic mode can combine to, e.x. circular polarization
\[
\frac{dE}{dt} = -L_{GW} < 0
\]

Photo credit: http://www.srl.caltech.edu/lisa/graphics/LISA_science.html
Why not detect gravitational tidal force instead?

Consider a spring with initial length \( \Delta r \) and characteristic frequency \( \omega = \frac{k}{m} \):

\[
(\Delta r)h(\frac{k}{m}) = (\Delta r)h\omega^2
\]

where

\[
h \propto \frac{\dot{r}}{r} \sim \frac{1}{r} \frac{MR^2}{rT^2} = \frac{1}{r} MR^2 \omega^2 \quad \text{(dimensionless)}
\]

and

\[
\omega^2 R = \frac{M}{R^2}
\]

hence the Gravitational Wave Force: \((\Delta r)\frac{1}{r} MR^2 \omega^4 = \frac{\Delta r}{r} M^3 R^3\)

which is far larger than the Gravitational Tidal Force:

\[
\frac{M}{r^2} \bigg(\frac{M}{(r+\Delta r)^2} \sim \frac{M \cdot \Delta r}{r^2 \cdot r}
\]
Other possible sources for GW

### Sources
- Compact object oscillation (periodic)
- Rotation of NS (periodic)
- Orbiting of compact binary (periodic)
- Merge of compact stars (burst)
- Supernovae (burst)
- Early universe (stochastic)
- NS tidal disruption
- BH excitation
- Cosmic strings

### Associate physics
- Rich physics included Spin-spin, spin-orbital coupling
- Enable to observe really young universe for the first time
- NS e.o.s., Is it a source of GRB?
- BH normal modes
- This is over my head
PSR B1913+16

- Discovered by Russell Alan Hulse and Joseph Hooton Taylor, Jr. in 1974.
- They were awarded the 1993 Nobel Prize in Physics.
  - Mass of detected pulsar 1.441 M⊙
  - Mass of companion 1.387 M⊙
  - Orbital period 7.751939106 hr
  - Eccentricity 0.617131
\[ E = \frac{1}{2} M \omega^2 r^2 + \frac{1}{2} M \omega^2 r^2 - \frac{M^2}{2r} \sim M^\frac{5}{3} \omega^\frac{2}{3} \]

\[ \Rightarrow \log E \propto \frac{2}{3} \log \omega \]

\[ \Rightarrow \frac{1}{E} \frac{dE}{dt} = \frac{2}{3} \frac{1}{\omega} \frac{d\omega}{dt} = -\frac{2}{3} \frac{1}{P} \frac{dp}{dt} \]

\[ \frac{dp}{dt} = -\frac{3}{2} \frac{P}{E} \frac{dE}{dt} = -\frac{3}{2} \frac{P}{E} L_{GW} \sim 10^{-13} \]

where \[ E \sim M^\frac{5}{3} \omega^\frac{2}{3} \]

\[ L_{GW} \sim \left(\frac{M(2r)^2}{T^3}\right)^2 \sim (M(2r)^2 \omega^3)^2 \sim \left(\frac{M}{\omega}\right)^{10} \]

a more detailed calculation shows \[ \frac{dP}{dt} = -2.4 \times 10^{-12} \]

and the observed value shows \[ \frac{dP}{dt} = -2.3 \pm 0.22 \times 10^{-12} \sim O(10^{-5} \text{ sec/ yr}) \]
Conclusion

• Why GW is weak and why quadrupole radiation?
• What’s the maximal power can GW carries?
• Why GW force larger than gravitational tidal force?
• The power of order of magnitude estimation.
Prospect

- Tools for observing our universe:
  1. EM wave
  2. Gravitational wave
  3. Cosmic ray
  4. Neutrino

- To know more than “estimations” we need numerical relativity to deal with the strong field near the sources (we have to solve the Einstein field equation and get the space-time structure)
3+1 decomposition

• The EM analog:

\[ \nabla \cdot \mathbf{E} = 0 \quad \text{Constraints equations} \]

\[ \nabla \cdot \mathbf{B} = 0 \]

\[ \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B} \quad \text{Evolution equations} \]

\[ \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \mathbf{E} \]

• Two categories of problems in Numerical Relativity:
  – Initial value problem
  – Evolution problem